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THE BRAUER RING OF A FIELD

BY

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The topic of this work is the Brauer group of a commutative ring. Recent investigations have yielded generalizations of this invariant in several directions; herein we wish to propose another: that the Brauer group be viewed as a subgroup of the unit group of the "Brauer ring". In justification, we should expect this ring to store and yield information about separable algebras which the Brauer group does not, and we should hope to be able to recover the Brauer group from purely ring theoretic properties. One purpose of the present paper is to describe the extent to which these goals are achieved. Another is to show that, in a categorical sense, the ring we shall describe is best possible.

To obtain structural results for the Brauer ring, we will make use of the theory of Green-functors, especially of the Burnside ring. Our structure theory will be independent of the construction of the Brauer ring, thus giving promise for applications in other areas. So as to not get too lost in our categorical approach, we have omitted some straight-forward axiom checking proofs; they may be regarded as exercises.

In this paper we will restrict our attention to the field case, leaving the Brauer ring of more general algebraic objects to another time.

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1. The Brauer ring

The purpose of this section is to construct the Brauer ring, and to outline a few elementary consequences of this construction.

Let E/F be a (not necessarily finite) Galois extension of fields with Galois group G. Let SEP(E,F) be the category of separable F-algebras A, with the center of A (denoted $Z(A)$) isomorphic as F-algebras with a finite product of subfields of E, each subfield being finite dimensional over F. If the extension E/F is understood, we may abbreviate SEP(E, F) as SEP. Plainly, SEP is closed under the formation of algebra products. It is also closed under tensor

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products. The proof of this requires a well known result, which we just state (see [15, Lemma 18.1a]).

1.1 LEMMA. *Let K_1/F, K_2/F be finite separable field extensions contained in the finite Galois extension E/F. Let $G = Gal(E/F)$ and $H_i = Gal(E/K_i)$. Choose $\sigma_1, \ldots, \sigma_n \in G$ to obtain a double coset decomposition*:

$$G = \bigcup_{i=1}^{n} H_1 \sigma_i H_2.$$

Then $K_1 \otimes_F K_2 \simeq L_1 \dotplus \cdots \dotplus L_n$, where $L_i = K_1\sigma_i(K_2)$ is a finite separable subfield of E/F.

A generalization of this result will appear in another paper by the author [9]. Now to show closure under tensor products, let A, $B \in$ SEP, with

$$Z(A) \simeq K_1 \dotplus \cdots \dotplus K_m, \; Z(B) \simeq L_1 \dotplus \cdots \dotplus L_n,$$

with each K_i, L_j a finite separable extension of F. Since any pair K_i, L_j can be embedded in a finite Galois extension of F contained in E, it follows that

$$Z(A \otimes_F B) \simeq Z(A) \otimes_F Z(B) \simeq \prod_{i,j} K_i \otimes_F L_j,$$

which is in turn isomorphic with a finite product of subfields of E, by 1.1.

It follows that we may form the Grothendieck ring of the category SEP (see [1, pp. 344–47]), which we denote by $S(E, F)$. We denote the image of an object $A \in$ SEP(E, F) in $S(E, F)$ by $[A]$. The following proposition collects some basic facts about Grothendieck rings, as applied to $S(E, F)$.

1.2 PROPOSITION. (a) *Every element of $S(E, F)$ can be expressed in the form $[A] - [B]$, some A, $B \in$ SEP.*

(b) *For elements $[A],[B]$ in $S(E, F)$, $[A] + [B] = [A \dotplus B]$, and $[A][B] = [A \otimes_F B]$. Also $1_{S(E,F)} = [F]$.*

(c) *If A, $B \in$ SEP, then $[A] = [B]$ if and only if $A \simeq B$ as F-algebras.*

Proof. (a), (b) and the if part of (c) are direct consequences of the definitions. For the only if part of (c), suppose $[A] = [B]$. Then there is an algebra $C \in$ SEP with $A \dotplus C \simeq B \dotplus C$ as F-algebras. Now apply the uniqueness statement in Wedderburn's theorem. ■

Note that if A is a finite dimensional, simple F-algebra, with $Z(A)$ F-isomorphic to a subfield of E, then A is a central simple $Z(A)$-algebra, and $Z(A)$ is a finite separable F-algebra. Since central simple algebras are separable, transitivity of separability implies that A is a separable F-algebra, hence $A \in \mathrm{SEP}$. Thus any product $A_1 \dotplus \cdots \dotplus A_r$, with each A_i simple and $Z(A_i)$ isomorphic to a finite dimensional subfield of E, is in SEP. Conversely, by Wedderburn's theorem, any algebra in SEP has this form uniquely up to F-isomorphism. This discussion, together with 1.2, establishes the next result.

1.3 PROPOSITION. *As an abelian group, $S(E, F)$ is free on the set*

$$\{[A]\colon A \in \mathrm{SEP},\ A \text{ is simple}\}.$$

$S(E, F)$ is too large to be manageable. We need to factor by an ideal roughly generated by differences of Morita equivalent algebras. Obtaining this ideal is our next goal.

1.4 PROPOSITION. *There is a group endomorphism β of $S(E, F)$ such that if $A \simeq M_n(D)$ as F-algebras, where $D \in \mathrm{SEP}$ is a division algebra, then $\beta([A]) = [D]$. The image of β is the subgroup of $S(E, F)$ that is freely generated by the set $\{[D]\colon D \in \mathrm{SEP}$ is a division algebra$\}$. Moreover, for all $u, v \in S(E, F)$, we have the identities $\beta(\beta(u)) = \beta(u)$ and $\beta(\beta(u)\beta(v)) = \beta(uv)$.*

Proof. If $A \in \mathrm{SEP}$ is simple, then $A \simeq M_n(D)$, where D is a division algebra with $Z(A) \simeq Z(D)$. Thus $D \in \mathrm{SEP}$. Moreover, if $B \simeq M_m(D') \in \mathrm{SEP}$ with $A \simeq B$, then by Wedderburn's theorem, $D \simeq D'$. It follows from 1.3 that the correspondence $[A] \to [D]$ gives a well defined group endomorphism β of $S(E, F)$ such that $\beta([M_n(D)]) = [D]$. The statement regarding the image of β is clear. Since $\beta(\beta([M_n(D)])) = [D] = \beta([M_n(D)])$, it follows that $\beta(\beta(u)) = \beta(u)$, all $u \in S(E, F)$. For the final identity, let $A \simeq M_n(D)$, $B \simeq M_m(D')$ be in SEP, where D, D' are division algebras. Since $D \otimes_F D'$ is semisimple, we can write $D \otimes_F D' \simeq M_{n_1}(D_1) \dotplus \cdots \dotplus M_{n_r}(D_r)$. Then,

$$\begin{aligned} A \otimes_F B &\simeq (M_n(F) \otimes_F D) \otimes_F (M_m(F) \otimes_F D') \\ &\simeq M_{mn}(F) \otimes_F (D \otimes_F D') \\ &\simeq M_{mn}(F) \otimes_F \big(M_{n_1}(D_1) \dotplus \cdots \dotplus M_{n_r}(D_r)\big) \\ &\simeq M_{mnn_1}(D_1) \dotplus \cdots \dotplus M_{mnn_r}(D_r). \end{aligned}$$

Therefore,

$$\begin{aligned} \beta(\beta([A])\beta([B])) &= \beta([D \otimes_F D']) \\ &= [D_1] + \cdots + [D_r] \\ &= \beta([A \otimes_F B]) \\ &= \beta([A][B]). \end{aligned}$$

Hence, $\beta(\beta(u)\beta(v)) = \beta(uv)$ for all $u, v \in S(E, F)$. ∎

1.5 Corollary. *$\ker\beta$ is an ideal of $S(E, F)$. As an ideal it is generated by* $\{[M_n(F)] - [F]: n \geq 1\}$.

Proof. Suppose $u \in \ker\beta$ and $v \in S(E, F)$. Then

$$\beta(uv) = \beta(\beta(u)\beta(v)) = \beta(0) = 0,$$

thus $\ker\beta$ is an ideal. Let I be the ideal of $S(E, F)$ generated by

$$\{[M_n(F)] - [F]: n \geq 1\}.$$

Plainly, $I \subseteq \ker\beta$. Conversely if $A \simeq M_n(D) \in$ SEP, then

$$[A] - \beta([A]) = [D]([M_n(F)] - [F]) \in I.$$

Extending linearly, it follows that $[B] - \beta([B]) \in I$ for all $B \in$ SEP. This if $[A] - [B] \in \ker\beta$, then $\beta([A]) = \beta([B])$ so that

$$[A] - [B] = ([A] - \beta([A])) - ([B] - \beta([B])) \in I.$$

Hence $\ker\beta \subseteq I$. ∎

The factor ring $S(E, F)/\ker\beta$ is called the *Brauer ring* of the extension E/F. We denote this ring by $B(E, F)$. If $E = F_S$ is the separable algebraic closure of F, then we denote $S(F_S, F) = S(F)$ and $B(F_S, F) = B(F)$. $B(F)$ is the *Brauer ring* of the field F. Note that whenever $E \subseteq E'$ is an inclusion of Galois extensions of F, there is an induced inclusion of categories SEP(E, F) $\subseteq$ SEP(E', F), hence also of rings $S(E, F) \subseteq S(E', F)$, $B(E, F) \subseteq B(E', F)$. Since every finite Galois extension of F is contained in F_S, and F_S is the union (direct limit) of such extensions, we obtain

1.6 Proposition. *Let F be any field. Then as rings,*

$$S(F) = \bigcup_E S(E, F) = \varinjlim_{E} S(E, F),$$

and

$$B(F) = \bigcup_E B(E, F) = \varinjlim_{E} B(E, F)$$

where the union and the limit are over the directed set of all finite Galois extensions of F contained in F_S.

The point of this proposition is that it effectively reduces the structure theory of $B(F)$ down to the computation of $B(E, F)$ for a finite Galois extension E/F.

For notation, we let $\langle A \rangle = [A] + \ker \beta$ denote the image of an algebra $A \in \text{SEP}$ in $B(E, F)$.

1.7 PROPOSITION. *For any Galois extension E/F, $B(E/F)$ is free (as an abelian group) on the generating set $\{\langle D \rangle: D \in \text{SEP}$ is a division algebra$\}$. Moreover, if D, D' are division algebras in* SEP, *then $\langle D \rangle = \langle D' \rangle$ if and only if $D \simeq D'$ as F-algebras.*

Proof. Since $\beta^2 = \beta$, it follows that $S(E, F) \simeq \ker \beta \oplus \operatorname{im} \beta$. Therefore, the canonical isomorphism of abelian groups $B(E, F) \simeq \operatorname{im} \beta$, together with 1.4, imply the first statement. If $\langle D \rangle = \langle D' \rangle$, then $[D] - [D'] \in \ker \beta$, so $0 = \beta([D] - [D']) = [D] - [D']$. Thus $D \simeq D'$ as F-algebras, by 1.2(c). ∎

For any field F, we let $\text{Br}(F)$ denote its Brauer group, and for a central simple F-algebra A, we let $\{A\}$ denote its class in $\text{Br}(F)$. It follows from 1.7 that for any Galois extension E of F, the mapping $\{A\} \to \langle A \rangle$ from $\text{Br}(F)$ to $B(E, F)$ is a well defined monomorphism into the group of units of $B(E, F)$.

From this point we could proceed directly to the structure of $B(E, F)$; however, we prefer to follow a less direct route. We hope in the end that our methods will justify themselves, in simplifying arguments which would otherwise be a morass of technicalities, and in building the framework for generalizations to other areas of research.

2. The F-Burnside ring

In this section we construct a generalization of the Burnside ring of a finite group (see [4], [16], [17]). We then prove a few elementary results which will be essential to our later work.

Throughout our present discussion, G will denote a fixed finite group. A G-set is a finite set on which G acts from the left. The category of all finite G-sets will be denoted $\mathcal{G}$; its morphisms are set maps which commute with the action of G. Note that if S and T are G-sets, then the disjoint union $S \dot{\cup} T$ and the cartesian product $S \times T$ (with the diagonal action) are also G-sets. Thus the set of isomorphism classes of finite G-sets becomes a commutative semi-ring. The Grothendieck ring constructed from this semi-ring is called the *Burnside ring* of G; it will be denoted $A(G)$. Elements of $A(G)$ are formal differences $[S] - [T]$ where $S, T \in \mathcal{G}$. Moreover, $[S] + [T] = [S \dot{\cup} T]$ and $[S][T] = [S \times T]$.

Let $P = P(G)$ denote the set of all conjugacy classes of subgroups of G. For each $b \in P$, pick a representative H_b of b, and let S_b denote the transitive

G-set of cosets modulo H_b. Then $A(G)$ is free (as an abelian group) on the set $\{[S_a]:\ a \in P\}$, that is, $\{S_a:\ a \in P\}$ is a complete set of representatives of isomorphism classes of transitive G-sets.

Let F: $\mathscr{G} \to \mathrm{AM}$ be a contravariant functor, where AM denotes the category of abelian monoids (we use AM for convenience; our construction would go through with AM replaced by the category of semi-groups). For a map of G-sets α: $S \to T$, we shall denote

$$\alpha^0 = F(\alpha)\colon F(T) \to F(S).$$

Such a functor will be called *additive* if given any two G-sets S_1, S_2, with inclusions K_i: $S_i \to S_1 \dot{\cup} S_2$, the induced map

$$K_1^0 \times K_2^0\colon F(S_1 \dot{\cup} S_1) \to F(S_1) \times F(S_2)$$

is an isomorphism. For an additive functor F and elements $x \in F(S_1)$, $y \in F(S_2)$, we introduce the notation $x \dot{+} y$ to denote the unique element of $F(S_1 \dot{\cup} S_2)$ satisfying $K_1^0 \times K_2^0(x \dot{+} y) = (x, y)$. For the remainder of this section, F will denote a fixed additive contravariant functor.

For any G-set S, we form the category (G, S, F) as follows:

Objects: Triples (T, ϕ, x) where $T \in \mathscr{G}$, ϕ: $T \to S$ is a G-map, and $x \in F(T)$.

Morphisms: A morphism $(T, \phi, x) \to (V, \psi, y)$ is a G-map α: $T \to V$ such that $\phi = \psi\alpha$ and $\alpha^0(y) = x$.

Given (T, ϕ, x), (V, ψ, y) in (G, S, F), define $(T, \phi, x) \oplus (V, \psi, y)$ to equal

$$\left(T \dot{\cup} V,\ \phi \dot{\cup} \psi,\ x \dot{+} y\right).$$

The latter is an object of (G, S, F) since F is additive. It is routine to check that $\oplus$ is a categorical coproduct for (G, S, F).

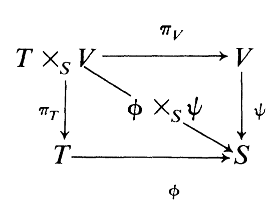
Given G-sets T and V together with G-maps ϕ: $T \to S$ and ψ: $V \to S$, the pullback of T and V over S is defined to be

$$T \times_S V = \{(t, v) \in T \times V\colon \phi(t) = \psi(v)\}.$$

There is a well defined G-map $\phi \times_S \psi$: $T \times_S V \to S$ given by

$$(\phi \times_S \psi)(t, v) = \phi(t)\ (= \psi(x)),$$

for all $(t, v) \in T \times_S V$, which makes the diagram



commute, where π_T, π_V are the projection mappings. With this notation we define $(T, \phi, x) \times_S (V, \psi, y)$ to equal $(T \times_S V, \phi \times_S \psi, \pi_T^0(x) \cdot \pi_V^0(y))$, which is again an object in (G, S, F).

The operations $\oplus$ and $\times_S$ satisfy all of the necessary identities to form the half ring $A_F^+(S)$ of isomorphism classes of objects in (G, S, F), with addition induced by $\oplus$ and multiplication by $\times_S$. We denote the associated Grothendieck ring by $A_F(S)$, and refer to this ring as the *F-Burnside ring* of G-sets over S. We let $[T, \phi, x]$ denote the image of (T, ϕ, x) in $A_F(S)$. The following lemma collects some standard results about this construction, as applied to $A_F(S)$.

2.1 LEMMA. (a) *Each element of $A_F(S)$ has the form $[T, \phi, x] - [V, \psi, y]$, for suitable $(T, \phi, x), (V, \psi, y)$ in (G, S, F).*
(b)

$$[T, \phi, x] + [V, \psi, y] = [T \dot{\cup} V, \phi \dot{\cup} \psi, x \dot{+} y],$$

and

$$[T, \phi, x][V, \psi, y] = [T \times_S V, \phi \times_S \psi, \pi_T^0(x) \cdot \pi_V^0(y)].$$

Moreover, $1_{A_F(S)} = [S, \mathrm{id}, 1_{F(S)}]$.
(c) $[T, \phi, x] = [V, \psi, y]$ *if and only if there exists (U, λ, z) in (G, S, F) such that $(T \dot{\cup} U, \phi \dot{\cup} \lambda, x \dot{+} z) \simeq (V \dot{\cup} U, \psi \dot{\cup} \lambda, y \dot{+} z)$ in (G, S, F).*

The next goal of this section is the strengthening of 2.1(c) above. We begin by considering the transitive case.

2.2 LEMMA. *Suppose (T, α, x), (V, β, y) and (W, γ, z) are in (G, S, F) and that T is a transitive G-set. If $(T, \alpha, x) \oplus (V, \beta, y) \simeq (T, \alpha, x) \oplus (W, \gamma, z)$ in (G, S, F), then $(V, \beta, y) \simeq (W, \gamma, z)$.*

Proof. By hypothesis, $(T \dot{\cup} V, \alpha \dot{\cup} \beta, x \dot{+} y) \simeq (T \dot{\cup} W, \alpha \dot{\cup} \gamma, x \dot{+} z)$, so there is a G-isomorphism

$$\phi: T \dot{\cup} V \to T \dot{\cup} W \quad \text{with } \alpha \dot{\cup} \beta = (\alpha \dot{\cup} \gamma)\phi \text{ and } \phi^0(x \dot{+} z) = x \dot{+} y.$$

Since T is transitive, and $\phi(T)$ is non-empty, either $\phi(T) = T$ or $\phi(T) \subseteq W$. We treat these cases separately.

Case (1) $\phi(T) = T$. Then $\phi(V) = W$. Write $\phi = \mu \dot{\cup} \lambda$, where

$$\mu = \phi|_T: T \to T \quad \text{and} \quad \lambda = \phi|_V: V \to W.$$

Let $K_V: V \to T \dot{\cup} V$, $K_W: W \to T \dot{\cup} W$ be inclusions. Clearly, $\phi K_V = K_W \lambda$, so that

$$\lambda^0(z) = \lambda^0 K_W^0(x \dot{+} z) = K_V^0 \phi^0(x \dot{+} z) = K_V^0(x \dot{+} y) = y.$$

Moreover, if $v \in V$, then

$$\gamma\lambda(v) = (\alpha \dot{\cup} \gamma)\phi(v) = (\alpha \dot{\cup} \beta)(v) = \beta(v),$$

that is, $\gamma\lambda = \beta$. Thus $\lambda\colon (V, \beta, y) \to (W, \gamma, z)$ is an isomorphism, finishing this case.

Case (2) $\phi(T) \subseteq W$, and therefore $T \subseteq \phi(V)$. Hence we may write

$$V = T_1 \dot{\cup} V' \quad \text{where} \quad \phi(T_1) = T,$$

and

$$W = T_2 \dot{\cup} W' \quad \text{where} \quad \phi(T) = T_2.$$

By additivity of F, write $y = x_1 \dot{+} y'$ and $z = x_2 \dot{+} z'$, where $x_i \in F(T_i)$, $y' \in F(V')$, and $z' \in F(W')$. We may also write $\phi = \mu \dot{\cup} \lambda \dot{\cup} \delta$, with

$$\mu = \phi|_T\colon T \to T_2, \quad \lambda = \phi|_{T_1}\colon T_1 \to T \quad \text{and} \quad \delta = \phi|_{V'}\colon V' \to W',$$

all isomorphisms. As in case 1, it follows that $\mu^0(x_2) = x$, $\lambda^0(x) = x_1$ and $\delta^0(z') = y'$. Define $\psi\colon V \to W$ to be $\mu\lambda \dot{\cup} \delta$. Then

$$\begin{aligned}\psi^0(z) &= (\mu\lambda \dot{\cup} \delta)^0(x_2 \dot{+} z') = (\mu\lambda)^0(x_2) \dot{+} \delta^0(z') \\ &= \lambda^0\mu^0(x_2) \dot{+} \delta^0(z') = x_1 \dot{+} y' = y.\end{aligned}$$

Finally, to show $\beta = \gamma\psi$, let $v \in V$. With the obvious inclusion mappings in mind we then have for $v \in T_1$,

$$\begin{aligned}\gamma\psi(v) &= \gamma\mu\lambda(v) = \gamma\phi\lambda(v) = (\alpha \dot{\cup} \gamma)\phi\lambda(v) = (\alpha \dot{\cup} \beta)\lambda(v) = \alpha\lambda(v) \\ &= (\alpha \dot{\cup} \gamma)\phi(v) = (\alpha \dot{\cup} \beta)(v) = \beta(v),\end{aligned}$$

while if $v \in V'$ then

$$\gamma\psi(v) = \gamma\delta(v) = (\alpha \dot{\cup} \gamma)\phi(v) = (\alpha \dot{\cup} \beta)(v) = \beta(v).$$

Thus, $\psi\colon (V, \beta, y) \to (W, \gamma, z)$ is an isomorphism. ∎

2.3 PROPOSITION. *Suppose* (T, α, x), (V, β, y), (W, γ, z) *are in* (G, S, F), *and satisfy* $(T, \alpha, x) \oplus (V, \beta, y) \simeq (T, \alpha, x) \oplus (W, \gamma, z)$. *Then* $(V, \beta, y) \simeq (W, \gamma, z)$.

Proof. Write $T = T_1 \dot\cup \cdots \dot\cup T_n$ as a disjoint union of transitive G-sets, and let $\alpha_i = \alpha|_{T_i}: T_i \to S$. By additivity of F, there exist $x_i \in F(T_i)$ so that $(T, \alpha, x) \simeq \oplus_{i=1}^n (T_i, \alpha_i, x_i)$. By 2.2, we may cancel the (T_i, α_i, x_i) one at a time, yielding the result. ∎

2.4 COROLLARY. *$[V, \beta, y] = [W, \gamma, z]$ in $A_F(S)$ if and only if $(V, \beta, y) \simeq (W, \gamma, z)$ in (G, S, F).*

We introduce some notational conveniences. If $H \leq G$, then G/H denotes the transitive G-set of left cosets modulo H. We will denote $A_F(G/H)$ by $A_F(H)$. If $H = G$, then for any non-empty G-set T, there is exactly one G-map $\eta_T: T \to G/G$. Thus we may abbreviate the category $(G, G/G, F)$ to (G, F), the object (T, η_T, x) of (G, F) to (T, x), and the element $[T, \eta_T, x]$ of $A_F(G)$ to $[T, x]$. Then isomorphism in (G, F) of objects (T, x) and (V, y) is equivalent with the existence of a G-isomorphism $\beta: T \to V$ with $\beta^0(y) = x$.

For any G-set T, let $W_T = \mathrm{Aut}_G(T)$. Especially, if $a \in P$ $(= P(G))$ we shall abbreviate W_{S_a} to $W_a = \mathrm{Aut}_G(S_a)$. We use W_T to define an equivalence relation $\sim_T$ on $F(T)$; namely, for elements $x, y \in F(T)$, we say $x \sim_T y$ if and only if there exists $\alpha \in W_T$ with $\alpha^0(x) = y$. For $a \in P$ we shall let $x \sim_a y$ denote $x \sim_{S_a} y$. For each $a \in P$, choose a set $R_a \subseteq F(S_a)$ of equivalence class representatives under $\sim_a$. With these conventions we are ready to complete the final task of this section: to show that $A_F(G)$ is free as an abelian group, and to explicitly display a basis.

2.5 LEMMA. *Fix $a \in P$ and suppose that $\sum_{i=1}^m [S_a, x_i] = \sum_{i=1}^n [S_a, y_i]$ for some $x_i, y_i \in R_a$. Then $m = n$, and there is a permutation π of $\{1, \ldots, n\}$ such that $x_i = y_{\pi(i)}$, all i.*

Proof. By 2.4,

$$\left(\dot{\bigcup_{i=1}^{m}} S_a, x_1 \dot+ \cdots \dot+ x_m\right) \simeq \left(\dot{\bigcup_{i=1}^{n}} S_a, y_1 \dot+ \cdots \dot+ y_n\right);$$

in particular, $\dot\cup_{i=1}^m S_a \simeq \dot\cup_{i=1}^n S_a$, so $m = n$. For notational ease, we set $S_a^i = S_a$, $1 \leq i \leq n$. Let $\alpha: \dot\cup_{i=1}^n S_a^i \to \dot\cup_{i=1}^n S_a^i$ be an isomorphism with

$$\alpha^0\left(y_1 \dot+ \cdots \dot+ y_n\right) = x_1 \dot+ \cdots \dot+ x_n.$$

For each i, $\alpha(S_a^i)$ is a transitive subset of $\dot\cup_{j=1}^n S_a^j$, so there is an index $\pi(i)$ with $\alpha(S_a^i) = S_a^{\pi(i)}$. This defines π. Since α is an isomorphism, π is a permutation of $\{1, \ldots, n\}$. For each i, let $K_i: S_a^i \to \dot\cup_{j=1}^n S_a^j$ be inclusion, and let

$$\alpha_i = \alpha|_{S_a^i}: S_a^i \to S_a^{\pi(i)}.$$

Plainly, $\alpha K_i = K_{\pi(i)}\alpha_i$. Thus

$$\begin{aligned}\alpha_i^0(y_{\pi(i)}) &= \alpha_i^0 K_{\pi(i)}^0(y_1 \dotplus \cdots \dotplus y_n)\\ &= K_i^0\alpha^0(y_1 \dotplus \cdots \dotplus y_n) = K_i^0(x_1 \dotplus \cdots \dotplus x_n) = x_i,\end{aligned}$$

that is, $x_i \sim_a y_{\pi(i)}$. Since $x_i, y_{\pi(i)} \in R_a$, it follows that $x_i = y_{\pi(i)}$, all i. ■

2.6. PROPOSITION. *Let F: $\mathcal{G} \to$ AM be an additive contravariant functor. Define $B_F = \{[S_a, x]: a \in P(G),\ x \in R_a\}$. Then B_F is a $\mathbf{Z}$-basis of $A_F(G)$.*

Proof. Let $[T, y] \in A_F(G)$. Write $T = \dot{\bigcup}_{i=1}^n T_i$, with each T_i a transitive G-set. By additivity of F, we may find elements $y_i \in F(T_i)$ with $[T, y] = \sum_{i=1}^n [T_i, y_i]$. For each i, choose $a_i \in P$ and an isomorphism α_i: $S_{a_i} \to T_i$. Then for each i, there is a unique $x_i \in R_{a_i}$ with $x_i \sim_{a_i} \alpha_{i_n}^0(y_i)$. Thus

$$(T_i, y_i) \cong (S_{a_i}, \alpha_i^0(y_i)) \cong (S_{a_i}, x_i)$$

so that $[T, y] = \sum_{i=1}^n [S_{a_i}, x_i]$, and B_F spans.

For independence, first suppose there is a dependence relation

$$\sum_{i=1}^n c_i[S_{a_i}, x_i] = 0$$

for some fixed $a \in P$, where $x_i \in R_a$ all i, and $x_i \neq y_i$ if $i \neq j$. Assume each c_i is non-zero. Then by 2.5, the equality

$$\sum_{c_i > 0} c_i[S_{a_i}, x_i] = \sum_{c_j < 0} (-c_j)[S_{a_j}, x_j]$$

yields $x_i = x_j$ for some $i \neq j$, a contradiction. In general, if there is a dependence relation $\sum_{a \in P}\sum_{x \in R_a} c_{a,x}[S_a, x] = 0$, then since the S_a are pairwise non-isomorphic, 2.4 yields $\sum_{x \in R_a} c_{a,x}[S_a, x] = 0$, for each $a \in P$. By the above argument, $C_{a,x} = 0$ for all $a \in P$, $x \in R_a$. ■

3. The structure of the F-Burnside algebra

Fixed in this section are a finite group G, and an additive functor F: $\mathcal{G} \to$ AM. By 2.6, $A_F(G)$ is torsion free (as an abelian group), and thus it embeds faithfully in the tensor product $\mathbf{Q} \otimes_Z A_F(G)$. For simplicity we shall denote $\mathbf{Q} \otimes_Z A_F(G)$ by $\mathbf{Q}A_F(G)$, and consider its elements to be rational multiples of elements of $A_F(G)$. The principal aim of this section is the explicit computation of $\mathbf{Q}A_F(G)$.

As may have become evident to the reader, there is an obvious isomorphism $A(G) \cong A_I(G)$, where I: $\mathcal{G} \to$ AM is the trivial functor. In particular, we may identify $A(G)$ with the subring of $A_F(G)$ consisting of the elements

$$\{[S,1] - [T,1]\colon S, T \in \mathscr{G}\}.$$

This is an important observation since, as we shall see, much of what is known about $\mathbf{Q}A(G)$ lifts up, in a suitably modified version, to all of $\mathbf{Q}A_F(G)$. First off then, we will recall some well known facts about the structure of $\mathbf{Q}A(G)$ (see [4], [5], [17]).

The set P has a natural partial ordering, where we set $a \leq b$ precisely when H_a is subconjugate to H_b (denoted $H_a \lesssim H_b$). Then $\mathbf{Q}A(G)$ has primitive idempotents $\{e_a\colon a \in P\}$, where $e_a = \sum_{b \leq a} \lambda_{b,a}[S_b]$ for suitable constants $\lambda_{b,a} \in \mathbf{Q}$. We set $\lambda_{b,a} = 0$ if $b \nleq a$ so that we may write

$$e_a = \sum_b \lambda_{b,a}[S_b].$$

It follows that $\sum_c e_c = 1_{A(G)}$, and that $e_a e_b = \delta_{ab} e_a$, for all $a, b \in P$.

For $a, b, c \in P$, let $V_{a,b,c}$ be the number of orbits in $S_a \times S_b$, under the diagonal action of G, which are isomorphic with S_c as G-sets. We summarize some known results on the constants $\lambda_{a,b}$ and $V_{a,b,c}$.

3.1 PROPOSITION. (a) *For $a, b \in P$, $[S_a][S_b] = \sum_c V_{a,b,c}[S_c]$. Thus the $V_{a,b,c}$ are structure constants for $A(G)$.*

(b) *For any $a \in P$, $V_{a,a,a} = \lambda_{a,a}^{-1} = [N_G(H_a)\colon H_a]$.*

(c) *For any $a, b, c \in P$, $V_{a,b,c} = 0$ unless both $c \leq a$ and $c \leq b$.*

(d) *For any $a \in P$, $|G| e_a \in A(G)$. Thus $|G| \cdot \lambda_{b,a} \in Z$ for all $a, b \in P$.*

We just remark that 3.1(d) can be strengthened to the statement

$$|N_G(H_a)| \cdot e_a \in A(G),$$

for any $a \in P$, by the idempotent formula of Gluck [7]. For brevity we shall denote $V_a = V_{a,a,a}$ and $V_{a,b} = V_{a,b,b}$, all $a, b \in P$. Fundamental to what follows are the following propositions relating the constants $V_{a,b,c}$ and $\lambda_{a,b}$.

3.2 PROPOSITION. *Let $a < b \in P$. Then for all $d \in P$, $\sum_{c \in P} \lambda_{c,b} V_{a,c,d} = 0$.*

Proof. Note that

$$\begin{aligned} 0 &= e_a \cdot e_b \\ &= \sum_{c,d} \lambda_{c,a} \lambda_{d,b} [S_c][S_d] \\ &= \sum_{c,d,e} \lambda_{c,a} \lambda_{d,b} V_{c,d,e} [S_e] \\ &= \sum_e \Big(\sum_{c,d} \lambda_{c,a} \lambda_{d,b} V_{c,d,e} \Big) [S_e]. \end{aligned}$$

It follows that

$$(*)\qquad \sum_{c,d} \lambda_{c,a}\lambda_{d,b}V_{c,d,e} = 0 \quad \text{for all} \quad e \in P.$$

We establish the required formula by induction on $a \in P$ with respect to the partial order $\leq$. If $a = \{1\}$ (the unique minimal element) then since $\lambda_{c,a} = 0$ if $c \nleq a$, $(*)$ becomes $\lambda_{a,a}\Sigma_d\lambda_{d,b}V_{a,d,e} = 0$, all $e \in P$. Since $\lambda_{a,a} \neq 0$ by 3.1(b), this starts the induction. Assume that $a \neq \{1\}$, and that whenever $c < a$, and $e \in P$, then $\Sigma_d\lambda_{d,b}V_{c,d,e} = 0$. By $(*)$, for any $e \in P$ we have

$$\begin{aligned} 0 &= \lambda_{a,a}\sum_d \lambda_{d,b}V_{a,d,e} + \sum_{c<a}\lambda_{c,a}\Big(\sum_d \lambda_{d,b}V_{c,d,e}\Big) \\ &= \lambda_{a,a}\sum_d \lambda_{d,b}V_{a,b,e} \quad \text{(by induction)}. \end{aligned}$$

Since $\lambda_{a,a} \neq 0$, $\Sigma_d\lambda_{d,b}V_{a,d,e} = 0$ for all $e \in P$, as claimed. ∎

3.3 PROPOSITION. *Let $a, b \in P$ with $b \nleq a$. Then $[S_a]e_b = 0$.*

Proof. Note that

$$\begin{aligned} [S_a]e_b &= [S_a]e_be_b \\ &= \sum_{c\leq b}\lambda_{c,b}[S_a][S_c]e_b \\ &= \sum_{c\leq b}\sum_{d\leq a,c}\lambda_{c,b}V_{a,c,d}[S_d]e_b. \end{aligned}$$

Thus it suffices to show $[S_d]e_b = 0$ whenever $c \leq b$ and $d \leq a, c$. The condition $b \nleq a$ then forces $d < b$, so we may as well assume $a < b$ to begin with. Then, by the above computation and 3.2,

$$[S_a]e_b = \sum_d\Big(\sum_c \lambda_{c,d}V_{a,c,d}\Big)[S_d]e_b = 0. \qquad ∎$$

3.4 PROPOSITION. (a) *For any $a \in P$, $e_a = V_a^{-1}[S_a]e_a$.*
(b) *If $a, c \in P$, then $\Sigma_b\lambda_{b,a}V_{a,b,c} = \lambda_{c,a}V_a$.*

Proof. (a)

$$\begin{aligned} e_a &= e_a \cdot e_a \\ &= \sum_{b\leq a}\lambda_{b,a}[S_b]e_a \\ &= \lambda_{a,a}[S_a]e_a \quad \text{(by 3.3)} \\ &= V_a^{-1}[S_a]e_a \quad \text{(by 3.1)}. \end{aligned}$$

(b) By (a),

$$\begin{aligned} e_a &= V_a^{-1}[S_a]e_a \\ &= V_a^{-1}\sum_b \lambda_{b,a}[S_a][S_b] \\ &= V_a^{-1}\sum_{b,c} \lambda_{b,c}V_{a,b,c}[S_c] \\ &= \sum_c \Big(V_a^{-1}\sum_b \lambda_{b,a}V_{a,b,c}\Big)[S_c]. \end{aligned}$$

Comparing coefficients yields $\lambda_{c,a} = V_a^{-1}\Sigma_b\lambda_{b,a}V_{a,b,c}$ as claimed. ■

Our next objective is to lift 3.3 up to the level of $A_F(G)$. We need a preliminary lemma that will be useful in other contexts as well.

3.5 LEMMA. *Let $a, b \in P$ and $x \in F(S_a)$. Then for some $r \geq 0$,*

$$[S_a, x][S_b, 1] = V_{a,b,a}[S_a, x] + \sum_{j=1}^{r}\left[S_{a_j}, x_j\right]$$

where $a_j < a$ and $x_j \in F(S_{a_j})$, $1 \leq j \leq r$.

Proof. Set $n = V_{a,b,a}$. If $a \nleq b$, then $n = 0$ and the result is clear. So we may assume $a \leq b$ and that $n > 0$. Set $S_a^i = S_a$, $1 \leq i \leq n$. Then by 3.1(c), there is an integer $r \geq 0$ so that

$$S_a \times S_b \cong S_a^1 \dot{\cup} \cdots \dot{\cup} S_a^n \dot{\cup} \dot{\bigcup_{j=1}^{r}} S_{a_j} = S,$$

for some $a_j < a$, $1 \leq j \leq r$. Let α: $S \to S_a \times S_b$ be this isomorphism, and let K_i: $S_a^i \to S$, l_j: $S_{a_j} \to S$ be the canonical injections. Let π: $S_a \times S_b \to S_a$ be the projection map. Since each composite $\pi\alpha K_i$: $S_a^i \to S_a$ is a G-map, it must be an automorphism, by the transitivity of S_a. Thus

$$[S_a, x] = \left[S_a^i, (\pi\alpha K_i)^0(x)\right] \in A_F(G), \quad \text{all } i.$$

Set $x_j = (\pi\alpha l_j)^0(x) \in F(S_{a_j})$. By the additivity of F, and the above remarks,

$$\begin{aligned} [S_a, x][S_b, 1] &= \left[S_a \times S_b, \pi^0(x)\right] \\ &= \left[S, \alpha^0\pi^0(x)\right] \\ &= \sum_{i=1}^{n}\left[S_a^i, K_i^0\alpha^0\pi^0(x)\right] = \sum_{j=1}^{r}\left[S_{a_j}, l_j\alpha^0\pi^0(x)\right] \\ &= \sum_{i=1}^{n}[S_a, x] + \sum_{j=1}^{r}\left[S_{a_j}, x_j\right] \\ &= V_{a,b,a}[S_a, x] + \sum_{j=1}^{r}\left[S_{a_j}, x_j\right]. \end{aligned}$$

■

3.6 PROPOSITION. *Let $a, b \in P$ with $b \nleq a$, and let $x \in F(S_a)$. Then* $[S_a, x]e_b = 0$.

Proof. The proof proceeds by induction on $a \in P$ with respect to $\leq$. If $a = \{1\}$, then by 3.2 and 3.5,

$$[S_a, x]e_b = \sum_c \lambda_{c,b}[S_a, x][S_c, 1] = \Big(\sum_c \lambda_{c,b} V_{a,c,a}\Big)[S_a, x] = 0.$$

Assume $[S_c, y]e_b = 0$ whenever $c < a$ and $y \in F(S_c)$ (note that $b \nleq a$ implies $b \nleq c$). Then several applications of 3.5 yields

$$\begin{aligned}[S_a, x]e_b &= [S_a, x]e_b \cdot e_b \\ &= \sum_c \lambda_{c,b}[S_a, x][S_c, 1]e_b \\ &= \sum_c \lambda_{c,d}\left(V_{a,c,a}[S_a, x] + \sum_{j=1}^{r_c}\left[S_{a_{j,c}}, x_{j,c}\right]\right)e_b \\ &= \Big(\sum_c \lambda_{c,b} V_{a,c,a}\Big)[S_a, x]e_b + \sum_c \sum_{j=1}^{r_c} \lambda_{c,b}\left[S_{a_{j,c}}, x_{j,c}\right]e_b.\end{aligned}$$

Since each $a_{j,c} < a$, induction implies that all $[S_{a_{j,c}}, x_{j,c}]e_b = 0$, and therefore,

$$[S_a, x]e_b = \Big(\sum_c \lambda_{c,b} V_{a,c,a}\Big)[S_a, x]e_b.$$

The hypothesis $b \nleq a$ implies that either $a < b$ or $a \nleq b$. If $a < b$, then 3.2 implies $\sum_c \lambda_{c,b} V_{a,c,a} = 0$. If $a \nleq b$, then $a \nleq c$ for all $c \leq b$, so that $V_{a,c,a} = 0$, all $c \leq b$ by 3.1(c). Since $\lambda_{c,b} = 0$ if $c \nleq b$, it follows that $\sum_c \lambda_{c,b} V_{a,c,a} = 0$ in this case also. In either case, this implies $[S_a, x]e_b = 0$. ■

For the next lemma recall that for any $a \in P$, $\mathrm{Aut}_G(S_a) \cong N_G(H_a)/H_a$, in particular $|\mathrm{Aut}_G(S_a)| = V_a$.

3.7 LEMMA. *Let $a \in P$, and set $S_a^i = S_a$, $1 \leq i \leq V_a$. Say that*

$$\mathrm{Aut}_G(S_a) = \{\sigma_i\colon 1 \leq i \leq V_a\}.$$

For each i, define $\alpha_i\colon S_a^i \to S_a \times S_a$ by $\alpha_i(s) = (s, \sigma_i(s))$. Then there is a (possibly empty) set $\{a_j\colon 1 \leq j \leq n\} \subseteq P$ with each $a_j < a$, and an isomorphism

$$\alpha\colon S_a^1 \dot\cup \cdots \dot\cup S_a^{V_a} \dot\cup \dot{\bigcup_{j=1}^{n}} S_{a_j} \to S_a \times S_a$$

such that if $K_i\colon S_a^i \to S_a^1 \dot\cup \cdots \dot\cup S_a^{V_a} \dot\cup \dot\bigcup_{j=1}^n S_{a_j}$ *is inclusion, then* $\alpha_i = \alpha K_i$, *all* i.

Proof. Since each α_i is injective, and $\alpha_i \neq \alpha_j$ if $i \neq j$, the lemma is a direct consequence of the definition of V_a and 3.2.

Since F is a functor, there is a natural action of W_S ($= \operatorname{Aut}_G(S)$) on $F(S)$, for any G-set S, given by $\sigma \cdot x = (\alpha^{-1})^0(x)$, all $x \in F(S)$, $\sigma \in W_S$. Contravariance implies $(\sigma\tau) \cdot x = \sigma \cdot \tau \cdot x$. For brevity we denote $\sigma \cdot x$ by x_σ. This action plays a key role in the structure of $A_F(G)$, as illustrated by the following proposition.

3.8 Proposition. *Let* $a \in P$, $x, y \in F(S_a)$. *Then*

$$[S_a, x][S_a, y]e_a = \sum_{\sigma \in W_a} [S_a, xy_\sigma]e_a.$$

Proof. Let $\{a_j\colon 1 \le j \le n\} \subseteq P$, $\alpha, \alpha_i, \sigma_i, K_i$ be as in Lemma 3.7. Let

$$S = S_a^1 \dot\cup \cdots \dot\cup S_a^{V_a} \dot\cup \dot{\bigcup_{j=1}^{n}} S_{a_j},$$

and let $\pi_i\colon S_a \times S_a \to S_a$ be the coordinate projection, $i = 1, 2$. Using the additivity of F, together with 3.6 and 3.7 we have

$$\begin{aligned}
[S_a, x][S_a, y]e_a &= \left[S_a \times S_a, \pi_1^0(x) \cdot \pi_2^0(y)\right]e_a \\
&= \left[S, \alpha^0\left(\pi_1^0(x) \cdot \pi_2^0(y)\right)\right]e_a \\
&= \sum_{i=1}^{V_a} \left[S_a^i, K_i^0\alpha^0\left(\pi_1^0(x) \cdot \pi_2^0(y)\right)\right]e_a \\
&= \sum_{i=1}^{V_a} \left[S_a^i, \alpha_i^0\left(\pi_1^0(x) \cdot \pi_2^0(y)\right)\right]e_a \\
&= \sum_{i=1}^{V_a} \left[S_a^i, (\pi_1\alpha_i)^0(x) \cdot (\pi_2\alpha_i)^0(y)\right]e_a \\
&= \sum_{i=1}^{V_a} \left[S_a^i, x \cdot (\sigma_i)^0(y)\right]e_a \\
&= \sum_{\sigma \in W_a} [S_a, xy_\sigma]e_a. \qquad \blacksquare
\end{aligned}$$

For any monoid H, let $\mathbf{Q}H$ denote the rational group algebra. For $a \in P$, define

$$\psi_a\colon \mathbf{Q}F(S_a) \to \mathbf{Q}A_F(G)e_a$$

by $\psi_a(x) = V_a^{-1}[S_a, x]e_a$, all $x \in F(S_a)$; then extend linearly to all of $\mathbf{Q}F(S_a)$.

3.9 Lemma. *For any $a \in P$, ψ_a is a surjective* **Q**-*space homomorphism.*

Proof. Everything is clear except surjectivity. It is sufficient to show that for any $b \in P$, $x \in F(S_b)$, we have $[S_b, x]e_a \in \operatorname{im} \psi_a$. We proceed by induction on b with respect to $\leq$. First note that if $a \nleq b$, then $[S_b, x]e_a = 0 \in \operatorname{im} \psi_a$, by 3.6, and if $a = b$, then $[S_b, x]e_a = \psi_a(V_a x) \in \operatorname{im} \psi_a$. In particular this covers the case $b = 1$, and we may assume $a < b$. Assume that $b > 1$, and that whenever $c < b$ and $y \in F(S_c)$, then $[S_c, y]e_a \in \operatorname{im} \psi_a$. Applying 3.4(a) and 3.5 we have

$$\begin{aligned}[S_b, x]e_a &= [S_b, x]e_a e_a \\ &= V_a^{-1}[S_b, x][S_a, 1]e_a \\ &= V_a^{-1}V_{b,a,b}[S_b, x]e_a + V_a^{-1}\sum_{j=1}^{r}\left[S_{b_j}, x_j\right]e_a\end{aligned}$$

where $b_j < b$, and $x_j \in F(S_{b_j})$, $1 \leq j \leq r$. Since $a < b$, 3.1(c) implies $V_{b,a,b} = 0$. By induction, each $[S_{b_j}, x_j]e_a \in \operatorname{im} \psi_a$, so that

$$[S_a, x]e_a = V_a^{-1}\sum_{j=1}^{r}\left[S_{b_j}, x_j\right]e_a \in \operatorname{im} \psi_a,$$

completing the induction step. ∎

If $S \in \mathcal{G}$ and $\sigma \in W_S$, then clearly $\sigma \cdot (xy) = (\sigma \cdot x)(\sigma \cdot y)$, all $x, y \in F(S)$. It follows that W_S acts as a group of ring automorphisms on $\mathbf{Q}F(S)$. We let $\mathbf{Q}F(S)^{W_S}$ denote the fixed ring under this action, that is,

$$\mathbf{Q}F(S)^{W_S} = \{x \in \mathbf{Q}F(S)\colon \sigma \cdot x = x \text{ all } \sigma \in W_S\}.$$

Then there is a **Q**-space epimorphism $\rho\colon \mathbf{Q}F(S) \to \mathbf{Q}F(S)^{W_S}$ given by

$$\rho(x) = |W_S|^{-1} \cdot \sum_{\sigma \in W_S} \sigma \cdot x.$$

Note that the restriction of ρ to $\mathbf{Q}F(S)^{W_S}$ is the identity; moreover, $\rho(x\rho(y)) = \rho(x)\rho(y)$, all $x, y \in \mathbf{Q}F(S)$. If $a \in P$ and $S = S_a$, then we let $\rho_a = \rho$, so that $\rho_a(x) = V_a^{-1}\Sigma_{\sigma \in W_a}\sigma \cdot x$, all $x \in \mathbf{Q}F(S_a)$. For convenience in what follows, we let χ_a denote the restriction of ψ_a to $\mathbf{Q}F(S_a)^{W_a}$.

3.10 PROPOSITION. *Let $a \in P$. Then $\psi_a = \psi_a \rho_a$. Moreover, χ_a is a surjective* **Q**-*algebra homomorphism.*

Proof. If $\sigma \in W_a$ and $x \in F(S_a)$, then $[S_a, x] = [S_a, x_\sigma]$, so that $\psi_a(x) = \psi_a(x_\sigma)$. Therefore, $\psi_a\rho_a(x) = V_a^{-1}\Sigma_{\sigma \in W_a}\psi_a(x_\sigma) = V_a^{-1}\Sigma_{\sigma \in W_a}\psi_a(x) = \psi_a(x)$. The first result follows, since $F(S_a)$ spans $\mathbf{Q}F(S_a)$. Furthermore, the surjectivity of ψ_a, together with $\psi_a = \psi_a\rho_a$, imply that χ_a is surjective. To see that χ_a is an algebra homomorphism, let $x, y \in F(S_a)$. Then

$$\begin{aligned}
\chi_a(\rho(x)\rho(y)) &= \chi_a(\rho(x \cdot \rho(y))) \\
&= \chi_a(x \cdot \rho(y)) \\
&= V_a^{-1} \sum_{\sigma \in W_a} \psi_a(xy_\sigma) \\
&= V_a^{-2} \sum_{\sigma \in W_a} [S_a, xy_\sigma]e_a \\
&= V_a^{-2}[S_a, x][S_a, y]e_a \quad \text{(by 3.8)} \\
&= \left(V_a^{-1}[S_a, x]e_a\right)\left(V_a^{-1}[S_a, y]e_a\right) \\
&= \psi_a(x) \cdot \psi_a(y) = \chi_a(\rho(x)) \cdot \chi_a(\rho(y)).
\end{aligned}$$

Since the elements $\{\rho(x) \colon x \in F(S_a)\}$ span $\mathbf{Q}F(S_a)^{W_a}$, χ_a is a **Q**-algebra homomorphism, as asserted. ∎

Of course our objective is to show that each χ_a is an isomorphism. This will follow from the next lemma.

3.11 LEMMA. *Let $x_1, \ldots, x_n \in F(S_a)$ be pairwise inequivalent (under $\sim_a$). Then $\{[S_a, x_i]e_a \colon 1 \le i \le n\}$ is a linearly independent subset of $\mathbf{Q}A_F(G)e_a$.*

Proof. For any i, 3.5 implies that

$$\begin{aligned}
[S_a, x_i]e_a &= \sum_{b \le a} \lambda_{b,a}[S_a, x_i][S_b, 1] \\
&= \lambda_{a,a}[S_a, x_i][S_a, 1] + \sum_{b<a} \lambda_{b,a}[S_a, x_i][S_b, 1] \\
&= [S_a, x_i] + \sum_j c_j\left[S_{a_j}, y_j\right],
\end{aligned}$$

some $a_j < a \in P$, $c_j \in \mathbf{Q}$, $y_j \in F(S_{a_j})$. Thus a dependence relation

$$\sum_{i=1}^{n} d_i[S_a, x_i]e_a = 0 \quad (d_i \in \mathbf{Q})$$

yields a dependence relation $\Sigma_{i=1}^{n} d_i[S_a, x_i] = 0$, by virtue of 2.4. But then the assumption on the x_i, together with 2.6, imply that $d_i = 0$, all i. ∎

3.12 THEOREM. *For any $a \in P$, the map χ_a: $\mathbf{Q}F(S_a)^{W_a} \to \mathbf{Q}A_F(G)e_a$ is a $\mathbf{Q}$-algebra isomorphism.*

Proof. All that remains is injectivity. Let R_a be a set of representatives for $\sim_a$ in $F(S_a)$. Let $x, y \in F(S_a)$, and suppose that $x \sim_a y$. Thus there is some $\alpha \in W_a$ with $\alpha^0(x) = y$. Then

$$\rho_a(y) = V_a^{-1} \sum_{\sigma \in W_a} \sigma^0(y) = V_a^{-1} \sum_{\sigma \in W_a} \sigma^0 \alpha^0(x)$$
$$= V_a^{-1} \sum_{\sigma \in W_a} (\alpha\sigma)^0(x) = V_a^{-1} \sum_{\sigma \in W_a} \sigma^0(x) = \rho_a(x).$$

It follows that $\{\rho_a(x): x \in R_a\}$ spans $\mathbf{Q}F(S_a)^{W_a}$ as a $\mathbf{Q}$-space. By 3.11, the set

$$\{\chi_a \rho_a(x): x \in R_a\} = \{V_a^{-1}[S_a, x]e_a: x \in R_a\}$$

is linearly independent over $\mathbf{Q}$. The result follows. ∎

3.13 THEOREM. *Let G be a finite group, and let F: $\mathcal{G} \to$ AM be an additive contravariant functor. Then there is a $\mathbf{Q}$-algebra isomorphism*

$$\mathbf{Q}A_F(G) \cong \prod_{a \in P} \mathbf{Q}F(S_a)^{W_a}.$$

Of course, the isomorphism in question is the product of the injections $\{\chi_a: a \in P\}$. Several ring theoretic properties of $\mathbf{Q}A_F(G)$ now become transparent. We single out the following.

3.14 COROLLARY. *If the functor F takes values in abelian groups, then $J(\mathbf{Q}A_F(G)) = 0$.*

Proof. By a result of Montgomery [13], if R is any ring acted upon by a finite group W of ring automorphisms, and if $|W|^{-1} \in R$, then $J(R^W) = J(R) \cap R^W$ (J denotes the Jacobson radical). Applying this to $R = \mathbf{Q}F(S_a)$ and $W = W_a$, it follows that $J(\mathbf{Q}F(S_a)^{W_a}) = 0$ (see [14, p. 73]). Since the radical respects products of rings, the result follows directly from 3.13. ∎

3.15 COROLLARY. *Suppose that for all $S \in \mathcal{G}$, $F(S)$ is a torsion abelian group. Suppose further that for every transitive G-set S, the action of W_S on $F(S)$ is trivial (every element of W_S acts as the identity of $F(S)$). Then $\mathbf{Q}A_F(G)$ is a von Neumann regular ring.*

Proof. Under the conditions imposed on F, it follows from a theorem of Villamayor [18] that $\mathbf{Q}F(S)$ is von Neumann regular for any $S \in \mathscr{G}$. The corollary follows since $\mathbf{Q}A_F(G) \cong \prod_a \mathbf{Q}F(S_a)$. ■

We shall finish this section by outlining a method by which the structure of $\mathbf{Q}A_F(G/H)$ can be computed for any subgroup $H \leq G$. These results will not be needed later, so details will be sparse at times. Throughout this discussion we fix a subgroup H of G.

If S is any H-set, then the *fibered product* of G with S, denoted $G \times_H S$, is the G-set of (equivalence classes of) pairs (g, s), where $g \in G$, $s \in S$, with the identifications $(g, s) = (gh^{-1}, hs)$, all $h \in H$, where the G-action on $G \times_H S$ arises from left multiplication in the first component. Given two H-sets S and T, and an H-map ϕ: $S \to T$, the map $\bar{\phi}$: $G \times_H S \to G \times_H T$ given by $\bar{\phi}(g, s) = (g, \phi(s))$ is a well defined G-map. It is easily seen that the correspondences $S \to G \times_H S$, $\phi \to \bar{\phi}$, define a covariant, sum preserving functor from $\mathscr{H}$ to $\mathscr{G}$.

Thus we may obtain from the additive functor F: $\mathscr{G} \to \mathrm{AM}$, an additive functor F_H: $\mathscr{H} \to \mathrm{AM}$, defined by $F_H(S) = F(G \times_H S)$, all $S \in \mathscr{H}$. For any H-map ϕ: $S \to T$, we have $F_H(\phi) = F(\bar{\phi})$: $F_H(T) \to F_H(S)$. We can now state the result of interest.

3.16 PROPOSITION. *Let F: $\mathscr{G} \to \mathrm{AM}$ be an additive contravariant functor, and let $H \leq G$. Then there is a ring isomorphism $A_F(G/H) \cong A_{F_H}(H/H)$.*

Proof. We shall define this isomorphism, and leave the rest as an exercise. Suppose S is a G-set, and α: $S \to G/H$ is a G-map. Let

$$S_\alpha = \{x \in S \colon \alpha(x) = 1H\}.$$

Plainly, S_α is an H-set. Denote by μ_α the G-map $G \times_H S_\alpha \to S$ given by $\mu_\alpha(g, s) = g \cdot s$. It follows easily that μ_α is a G-isomorphism. Then the isomorphism in question is given by Λ: $A_F(G/H) \to A_{F_H}(H/H)$, where

$$\Lambda([S, \alpha, x]) = [S_\alpha, \mu_\alpha^0(x)], \quad \text{all } [S, \alpha, x] \in A_F(G/H). \qquad ■$$

The reader can now deduce several corollaries by combining this result with 3.13, 3.14 and 3.15.

4. The structure of the Brauer algebra

In this section we shall connect the Brauer ring with the F-Burnside ring, for suitably chosen F. The tool at our disposal is Galois theory. However, Galois theory only connects intermediate subfields with subgroups (hence with transitive G-sets), whereas our needs will require connecting finite products of intermediate subfields with G-sets. The details of this extension are messy, but straightforward. The author could not find these results, although they are alluded to in [5].

We will not be working with arbitrary additive functors in this section, so without confusion let E/F denote a finite Galois extension of fields, with Galois group $G = Gal(E/F)$. The category $\mathcal{G}$ of finite G-sets is then anti-equivalent with the category CSEP(E, F) of commutative algebras in SEP(E, F). This anti-equivalence is given as follows. For $S \in \mathcal{G}$, define $R_S = \operatorname{Hom}_G(S, E)$, under pointwise operations. Then $R_S \in$ CSEP. Moreover, if $S \cong G/H$ for some subgroup H of G, then $R_S \cong E^H$ (fixed field of H) under the correspondence $\gamma \to \gamma(1H)$. For a G-map ϕ: $S \to T$, there is an induced F-algebra homomorphism ϕ_*: $R_T \to R_S$, given by $\phi_*(\gamma) = \gamma\phi$, all $\gamma \in R_T$. Conversely, if $R \in$ CSEP, define $S_R = \operatorname{Hom}_F(R, E)$, a finite set, which becomes a G-set using the G-action on E. Again note that if L is a subfield of E/F, then S_L is isomorphic with the transitive G-set of cosets modulo $Gal(E, L)$. If α: $R \to R'$ is an F-algebra homomorphism, then the map α^*: $S_{R'} \to S_R$, given by $\alpha^*(f) = f \circ \alpha$, is a G-map. The details of this anti-equivalence we shall need are summarized in the following sequence of lemmas (complete proofs are given in [8]).

4.1 LEMMA. *Let S and T be any G-sets, and suppose α: $R_T \to R_S$ is an F-algebra isomorphism. Then there is a G-isomorphism ϕ: $S \to T$ such that $\phi_* = \alpha$.*

4.2 LEMMA. *Suppose α, β: $S \to T$ are G-maps, with T a transitive G-set. If $\alpha_* = \beta_*$: $R_T \to R_S$, then $\alpha = \beta$.*

4.3 LEMMA. *If S_1, S_2 are G-sets, then $R_{S_1 \dot\cup S_2} \cong R_{S_1} \dotplus R_{S_2}$. This isomorphism takes $\alpha \in R_{S_1 \dot\cup S_2}$ to the pair $(\alpha | S_1, \alpha | S_2)$.*

The next lemma says that pullbacks in $\mathcal{G}$ correspond to tensor products in CSEP.

4.4 LEMMA. *Let $H \leq G$. Let S_1, S_2 be G-sets, and suppose there are G-maps α_i: $S_i \to G/H$, $i = 1,2$. Denote the pullback: $S_1 \times_{G/H} S_2$ by T, and let $K = R_{G/H}$. Define Φ: $R_{S_1} \otimes_K R_{S_2} \to R_T$ by $\Phi(f \otimes g)(x, y) = f(x)g(y)$. Then Φ is an $R_{G/H}$-algebra and $R_{S_1} - R_{S_2}$ bimodule isomorphism.*

The reason for the previous discussion is the following observation. If ρ: CSEP $\to$ AM is any covariant, product preserving functor, then we may construct an additive contravariant functor F_ρ: $\mathcal{G} \to$ AM by defining $F_\rho(S) = \rho(R_S)$, and for a G-map ϕ: $S \to T$, we let $F_\rho(\phi) = \rho(\phi_*)$: $F_\rho(T) \to F_\rho(S)$. As usual, we also denote $\phi^0 = F_\rho(\phi)$. It follows that we may obtain the F_ρ-Burnside rings $A_{F_\rho}(S)$, $S \in \mathcal{G}$, which for brevity we denote $A_\rho(S)$.

Our applications arise as follows. For any commutative ring R, let $\mathrm{AZ}(R)$ denote the category of Azumaya (central separable) R-algebras. When R is a field, $\mathrm{AZ}(R)$ coincides with the category of finite dimensional, central simple R-algebras. For an algebra A in $\mathrm{AZ}(R)$, let (A) denote its R-algebra isomorphism class, and $\{A\}$ denote its image in the Brauer group, $\mathrm{Br}(R)$. Denote the set of all isomorphism classes in $\mathrm{AZ}(R)$ by $\mathrm{AZ}_0(R)$. Then $\mathrm{AZ}_0(R)$ becomes an abelian monoid under tensor products over R, with identity element (R). If ϕ: $R \to S$ is a homomorphism of commutative rings, then the correspondence $(A) \to (S \otimes_R A)$ (where S is considered an R-algebra via ϕ) defines a monoid homomorphism $\mathrm{AZ}_0(R) \to \mathrm{AZ}_0(S)$. Moreover, for any commutative rings R, S, there is a natural isomorphism

$$\mathrm{AZ}_0(R \dotplus S) \cong \mathrm{AZ}_0(R) \times \mathrm{AZ}_0(S).$$

Similar remarks apply to $\mathrm{Br}(R)$. Thus the correspondences $R \to \mathrm{AZ}_0(R)$, $R \to \mathrm{Br}(R)$ yield covariant, product preserving functors from CSEP to AM. By the construction of the previous paragraph, we obtain the rings $A_{\mathrm{AZ}}(S)$, and $A_{\mathrm{Br}}(S)$, for any G-set S. More explicitly, a typical element of $A_{\mathrm{AZ}}(S)$ will be a formal difference

$$[T_1, \phi_1, (A_1)] - [T_2, \phi_2, (A_2)],$$

where T_i is a G-set, ϕ_i: $T_i \to S$ is a G-map, and $(A_i) \in \mathrm{AZ}(R_{T_i})$, $i = 1, 2$. A similar description holds for $A_{\mathrm{Br}}(S)$.

We need one last bit of notation before we can attack the main result of this section. Let $H \leq G$, $S \in G$, and suppose α: $S \to G/H$ is a G-map. If $A \in \mathrm{AZ}(R_S)$, then define the $R_{G/H}$-algebra A_α to be A as a ring, with $R_{G/H}$ action induced from α_*: $R_{G/H} \to R_S$. Thus, for $x \in R_{G/H}$ and $a \in A$, we have $x \cdot a = \alpha_*(x)a$. Note that $A \cong A_\alpha$ as F-algebras, since α_* is an F-algebra homomorphism.

4.5 LEMMA. *Let $H \leq G$, and let $[S, \alpha, (A)]$, $[T, \beta, (B)]$ be in $A_{\mathrm{AZ}}(H)$. Then $[S, \alpha, (A)] = [T, \beta, (B)]$ if and only if $A_\alpha \cong B_\beta$ as $R_{G/H}$-algebras.*

Proof. ($\Rightarrow$) By 2.4, there is a G-isomorphism ϕ: $T \to S$ with $\alpha\phi = \beta$ and $\phi^0((A)) = (B)$. This last condition yields an R_T-algebra isomorphism

$$\psi\colon R_T \otimes_{R_S} A \to B.$$

Define γ: $A \to B$ by $\gamma(a) = \psi(1 \otimes a)$, all $a \in A$. Since $R_S \cong R_T$, γ is a ring isomorphism. Furthermore, if $x \in R_{G/H}$, then

$$\begin{aligned}\gamma(x \cdot a) &= \gamma(\alpha_*(x)a) = \psi(1 \otimes \alpha_*(x)a) = \psi(\phi_*\alpha_*(x) \otimes a) \\ &= \psi(\beta_*(x) \otimes a) = \beta_*(x)\psi(1 \otimes a) = \beta_*(x)\gamma(a) = x \cdot \gamma(a).\end{aligned}$$

Thus, γ is an $R_{G/H}$-algebra isomorphism of A_α to B_β.

($\Leftarrow$) Suppose $\gamma\colon A_\alpha \to B_\beta$ is an $R_{G/H}$ algebra isomorphism. Then $\gamma(Z(A_\alpha)) = Z(B_\beta)$, that is, $\gamma(R_S) = R_T$. By 4.1, there is a G-isomorphism $\phi\colon T \to S$ with $\phi_* = \gamma$. We claim that $\alpha\phi = \beta$. By 4.2, it is enough to show that

$$\beta_* = \phi_*\alpha_*\colon R_{G/H} \to R_T.$$

If $x \in R_{G/H}$, then

$$\phi_*\alpha_*(x) = \gamma(\alpha_*(x)) = \beta_*(x)\gamma(1_A) = \beta_*(x).$$

Note that the map $\psi\colon R_T \otimes_{R_S} A \to B$, given by $\psi(x \otimes a) = x\gamma(a)$, is an R_T-algebra isomorphism, so that $\phi^0(A) = (B)$ in $\mathrm{AZ}_0(R_T)$. It follows that

$$\phi\colon (T, \beta, (B)) \to (S, \alpha, (A))$$

is an isomorphism. ■

For any subgroup $H \leq G$, the isomorphism $R_{G/H} \cong E^H$ allows us to replace $R_{G/H}$ by E^H. Define functions $\psi_H\colon A_{\mathrm{AZ}}(H) \to S(E, E^H)$ by

$$\psi_H([S, \alpha, (A)]) = [A_\alpha]$$

and $\chi_H\colon A_{\mathrm{Br}}(H) \to B(E, E^H)$ by

$$\chi_H([S, \alpha, \{A\}]) = \langle A_\alpha \rangle.$$

4.6 THEOREM. *Let E/F be a finite Galois extension with Galois group G, and let $H \leq G$. Then the functions ψ_H and χ_H are ring isomorphisms, that is,*

$$A_{\mathrm{AZ}}(H) \cong S(E, E^H) \quad \textit{and} \quad A_{\mathrm{Br}}(H) \cong B(E, E^H).$$

Proof. We give the proof for ψ_H, the proof for χ_H is almost identical. For convenience, let $\psi = \psi_H$. By 4.5, ψ is well defined and injective. Let $[S, \alpha, (A)], [T, \beta, (B)]$ be in $A_{\mathrm{AZ}}(H)$. Since

$$(A) \dot{+} (B) = (A \dot{+} B) \quad \text{and} \quad A_\alpha \dot{+} B_\beta = (A \dot{+} B)_{\alpha \dot{\cup} \beta},$$

we have

$$\begin{aligned} \psi([S, \alpha, (A)] + [T, \beta, (B)]) &= \psi\big([S \dot{\cup} T, \alpha \dot{\cup} \beta, (A \dot{+} B)]\big) \\ &= \big[(A \dot{+} B)_{\alpha \dot{\cup} \beta}\big] = [A_\alpha] + \big[B_\beta\big] \\ &= \psi([S, \alpha, (A)]) + \psi([T, \beta, (B)]), \end{aligned}$$

so ψ preserves sums. For products, let V denote the pullback $S \times_{G/H} T$, with projections π_S, π_T, and let $\phi: V \to G/H$ be the induced map ($\phi = \alpha\pi_S = \beta\pi_T$). Let $K = R_{G/H}$. Note that

$$\begin{aligned}
\pi_S^0(A) \cdot \pi_T^0(B) &= \left(R_V \otimes_{R_S} A\right) \cdot \left(R_V \otimes_{R_T} B\right) \\
&= \left(A \otimes_{R_S} R_V \otimes_{R_V} R_V \otimes_{R_T} B\right) \\
&= \left(A \otimes_{R_S} R_V \otimes_{R_T} B\right) \\
&= \left(A \otimes_{R_S} R_S \otimes_K R_T \otimes_{R_T} B\right) \quad \text{(by 4.4)}. \\
&= (A \otimes_K B).
\end{aligned}$$

Since the identity map $(A \otimes_K B)_\phi \to A_\alpha \otimes_K B_\beta$ is a K-algebra isomorphism, the above implies

$$\begin{aligned}
\psi([S, \alpha, (A)] \cdot [T, \beta, (B)]) &= \psi\left(\left[V, \phi, \pi_S^0(A) \cdot \pi_T^0(B)\right]\right) \\
&= \left[(A \otimes_K B)_\phi\right] \\
&= [A_\alpha]\left[B_\beta\right] \\
&= \psi([S, \alpha, (A)]) \cdot \psi([T, \beta, (B)]).
\end{aligned}$$

To see that ψ is surjective, let $A \in \mathrm{SEP}(E, E^H)$ be simple, with $Z(A) \cong E^J$ for some subgroup $J \leq H$. Let $\alpha: G/J \to G/H$ be projection, that is, $\alpha(gJ) = gH$, all $g \in G$. Then, viewing A as an $R_{G/J}$-algebra via the $R_{G/H}$-isomorphism $R_{G/J} \cong E^J$, we have $A \in \mathrm{AZ}(R_{G/J})$. An easy computation then yields $\psi([G/J, \alpha, (A)]) = [A]$, so that ψ is surjective, by 1.3. ∎

Let us combine this theory with 3.13 to obtain the structure of $\mathbf{Q}B(E, F)$ $(= \mathbf{Q} \otimes_Z B(E, F))$. For this recall that isomorphism classes of transitive G-sets correspond bijectively with conjugacy classes of subgroups of G, hence with F-isomorphism classes of intermediate subfields in the extension E/F. Let $S \cong G/H$ be a transitive G-set, and set $L = E^H$. Then $\mathrm{Aut}_G(S) \cong Gal(L/F)$. The only remaining detail is to describe the action of $Gal(L/F)$ on $\mathrm{Br}(L)$. It turns out that this action is well known (see [2], [3], [6], or [10]); however, for the readers convenience, we will be explicit.

For any $\sigma \in Gal(L/F)$ and central simple L-algebra A, define a new R-algebra A^σ by letting $A = A^\sigma$ as rings, with L-algebra structure given by $l * a = \sigma^{-1}(l)a$, where the multiplication on the right is that in A. Then, as in [3], or [10], it is seen that this induces a bona fide action of $Gal(L/F)$ on $\mathrm{Br}(L)$, given by $\sigma\{A\} = \{A^\sigma\}$. As before, this action extends to an action by ring automorphisms of the group algebra $\mathbf{Q}\,\mathrm{Br}(L)$. For convenience in stating the next theorem, we let $H_L = Gal(L/F)$.

4.7 THEOREM. *Let E/F be a finite Galois extension. Then*

$$\mathbf{Q}B(E,F) \cong \prod_L \mathbf{Q}\,\mathrm{Br}(L)^{H_L}$$

where the product is over a set of representatives of F-isomorphism classes of intermediate subfields of the extension E/F.

Proof. Follows directly from 3.13, 4.6, and the preceding remarks. ∎

We pause to give an example by computing the Brauer algebra of the p-adic field $\mathbf{Q}_P$, for any prime $0 \neq p \in \mathbf{Z}$. We quote a lemma due to Janusz [10]. Its essential content is that for p-adic fields, the action on the Brauer group that we have described is trivial.

4.8 LEMMA. *Let $0 \neq p \in \mathbf{Z}$ be a prime. For $i = 1, 2$, let L_i be a finite extension of $\mathbf{Q}_P$, and let A_i be a central simple L_i-algebra. If A_1 and A_2 are isomorphic as rings, then* $\mathrm{inv} A_1 = \mathrm{inv} A_2$.

We remark that the notation $\mathrm{inv} A$ for a central simple L-algebra A denotes its Hasse invariant. For a discussion of this invariant, see [15]. The most important fact for us is that the class of the algebra A in $\mathrm{Br}(L)$ is completely determined by its Hasse invariant. If $\sigma \in Gal(L/F)$, it follows since $A \cong A^\sigma$ as rings, that $\mathrm{inv} A^\sigma = \mathrm{inv} A$. Hence $\sigma\{A\} = \{A^\sigma\} = \{A\}$. Combining this with 4.7, and the well known fact that the Brauer group of a local field is $\mathbf{Q}/\mathbf{Z}$, we have the following.

4.9 THEOREM. *Let E be a finite Galois extension of the p-adic field $\mathbf{Q}_p$, and let n be the number of $\mathbf{Q}_p$-isomorphism classes of intermediate subfields of $E/\mathbf{Q}_p$. Then $\mathbf{Q}B(E, \mathbf{Q}_p) \cong \prod_n \mathbf{Q}(\mathbf{Q}/\mathbf{Z})$ where the right hand side is a product of n copies of the group algebra $\mathbf{Q}(\mathbf{Q}/\mathbf{Z})$.*

Passing to direct limits we can state the following.

4.10 COROLLARY. *The Brauer algebra $\mathbf{Q}B(\mathbf{Q}_p)$ is von Neumann regular.*

Proof. The result is direct from 1.6, 3.15, 4.9, and the fact that the property of being von Neumann regular is preserved under the taking of direct limits. ∎

Note that 4.8 holds trivially for the real numbers $\mathbf{R}$, so that by 4.7 we have

$$\mathbf{Q}B(\mathbf{C},\mathbf{R}) \simeq \mathbf{Q}(C_2) \dotplus \mathbf{Q} \simeq \mathbf{Q} \dotplus \mathbf{Q} \dotplus \mathbf{Q},$$

where C_2 is the cyclic group of order two. This isomorphism does not behave well at the level of $B(\mathbf{C},\mathbf{R})$; in fact, it can be shown that

$$(4\mathbf{Z})(C_2) \dot{+} 4\mathbf{Z} \subset B(\mathbf{C},\mathbf{R}) \subset \mathbf{Z}(C_2) \dot{+} \mathbf{Z},$$

with both containments proper. More generally, if F is an additive contravariant functor such that for every transitive G-set S the action of W_S on $F(S)$ is trivial (as in 3.15), then one has

$$\prod_{a\in P} (|G|^2\mathbf{Z})F(S_a) \subset A_F(G) \subset \prod_{a\in P} \mathbf{Z}F(S_a).$$

In particular, the quotient of the last two groups has $|G|^2$-torsion (see [8], Corollary 6.5).

The structure of $B(E/\mathbf{Q})$ is currently under study.

5. Functorial properties

The importance of the isomorphism 4.6 lies not only with its structural consequences, but also in the fact that the correspondence $S \to A_F(S)$ defines a functor with especially nice properties. It is the purpose of this section to describe these properties, and to show what consequences they have for associative algebras. We start with a definition (see [5] or [11]).

DEFINITION 5.1. Let G be a finite group, and let AB denote the category of abelian groups. A *Green-functor* on G is a bifunctor $M = (M^*, M_*)\colon \mathcal{G} \to \mathrm{AB}$, where M^* is covariant, M_* is contravariant, M^* and M_* agree on objects, such that the following conditions are fulfilled by M.

(a) If

$$\begin{array}{ccc} X & \xrightarrow{\chi_2} & X_2 \\ {\scriptstyle \chi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ X_1 & \xrightarrow[\phi_1]{} & Y \end{array}$$

is a pullback diagram in G, then the following diagram commutes:

$$\begin{array}{ccc} M(X) & \xrightarrow{M^*(\chi_2)} & M(X_2) \\ {\scriptstyle M_*(\chi_1)}\uparrow & & \uparrow{\scriptstyle M_*(\phi_2)}. \\ M(X_1) & \xrightarrow[M^*(\phi_1)]{} & M(Y) \end{array}$$

(b) If $S_1, S_2 \in \mathcal{G}$ with inclusions K_i: $S_i \to S_1 \dot{\cup} S_2$, then the homomorphisms

$$M_*(K_i)\colon M(S_1 \dot{\cup} S_2) \to M(S_i)$$

induce an isomorphism

$$M_*(K_1) \times M_*(K_2)\colon M(S_1 \dot{\cup} S_2) \to M(S_1) \times M(S_2).$$

(c) For each G-set S, $M(S)$ is a commutative ring with 1.

(d) For each G-map α: $S \to T$, $M_*(\alpha)$ is a ring homomorphism (preserving unit).

(e) For each G-map α: $S \to T$ we may view $M(S)$ as a $M(T)$-module via $M_*(\alpha)$. We then require $M^*(\alpha)$ to be a $M(T)$-module homomorphism. Thus, for any $s \in M(S)$, $t \in M(T)$, we have $M^*(\alpha)(M_*(\alpha)(t) \cdot s) = t(M^*(\alpha)(s))$.

Let F: $\mathcal{G} \to$ AM be a contravariant additive functor. We now show how to turn A_F into a Green-functor.

Suppose $S, T \in \mathcal{G}$, and α: $S \to T$ is a G-map. Then the map

$$\alpha^* = A_F^*(\alpha)\colon A_F(S) \to A_F(T)$$

given by

$$\alpha^*([V, \phi, x]) = [V, \alpha\phi, x]$$

is a group homomorphism. To describe a map $\alpha_* = A_{F*}(\alpha)$: $A_F(T) \to A_F(S)$, note that for any $[W, \psi, y] \in A_F(T)$, there is a pullback diagram

$$\begin{array}{ccc} W \times_T S & \xrightarrow{\pi_S} & S \\ {\scriptstyle \pi_W}\downarrow & & \downarrow{\scriptstyle \alpha} \\ W & \xrightarrow[\psi]{} & T \end{array} ,$$

hence we obtain the element $[W \times_T S, \pi_S, \pi_W^0(y)]$ of $A_F(S)$. We then define α_* by $\alpha_*([W, \psi, y]) = [W \times_T S, \pi_S, \pi_W^0(y)]$.

A tedious but routine check of the axioms in 5.1 establishes the following.

5.2 THEOREM. *Let G be a finite group, and let F: $\mathcal{G} \to$ AM be a contravariant additive functor. Then $A_F = (A_F^*, A_{F*})$ is a Green-functor.*

Because of this result, we should be able to locate analogues of A_F^*, A_{F*} in the Brauer ring. For notational convenience, let E/D denote a finite Galois extension. For any intermediate subfield $D \subseteq K \subseteq E$, we shall let $[A]_K$ (resp. $\langle A \rangle_K$) denote the image of $A \in$ SEP(E, K) in $S(E, K)$ (resp. $B(E, K)$).

5.3 PROPOSITION. *Let $D \subseteq K \subseteq L \subseteq E$ be a tower of fields, with E/D a finite Galois extension.*

(a) *There is a group homomorphism* $\text{ind} = \text{ind}_{L \to K}$: $S(E, L) \to S(E, K)$, *such that* $\text{ind}([A]_L) = [A]_K$ *for all* $A \in \text{SEP}(E, L)$.

(b) ind *factors through the projection of S to B, that is, there is a group homomorphism* $\overline{\text{ind}} = \overline{\text{ind}}_{L \to K}$: $B(E, L) \to B(E, K)$ *such that the following diagram commutes*:

$$\begin{array}{ccc} S(E,L) & \xrightarrow{\text{ind}} & S(E,K) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B(E,L) & \xrightarrow[\overline{\text{ind}}]{} & B(E,K) \end{array}$$

Proof. (a) Clear from 1.3.

(b) Let β_K denote the endomorphism of $S(E, K)$ given in 1.4. We must show that $\text{ind}_{L \to K}(\ker \beta_L) \subseteq \ker \beta_K$. Suppose $[A]_L - [B]_L \in \ker \beta_L$. Write

$$A \underset{L}{\cong} M_{n_1}(D_1) \dotplus \cdots \dotplus M_{n_r}(D_r) \quad \text{and} \quad B \underset{L}{\cong} M_{m_1}(D_1') \dotplus \cdots \dotplus M_{m_s}(D_s').$$

Then 1.2(c), together with the uniqueness statement of Wedderburns theorem, insures $r = s$, and (wlog) $D_i \cong D_i'$ as L-algebras, all i. Thus $D_i \cong D_i'$ as K-algebras, all i, so that $[A]_K - [B]_K \in \ker \beta_K$ since $A \cong A_\alpha$ as D-algebras. ■

5.4 PROPOSITION. *Let $D \subseteq K \subseteq L \subseteq E$ be a tower of fields, with E/D a finite Galois extension.*

(a) *There is a ring homomorphism* $\text{res} = \text{res}_{K \to L}$: $S(E, K) \to S(E, L)$ *such that* $\text{res}([A]_K) = [L \otimes_K A]_L$, *all* $A \in \text{SEP}(E, K)$.

(b) res *factors through the projection of S onto B.*

Proof. (a) The existence of res follows from 1.3 and the observation that if $A \cong B$ as K-algebras, then $L \otimes_K A \cong L \otimes_K B$ as L-algebras. Hence, from the distributive property of tensor products over algebra products, and the fact that $L \otimes_K (A \otimes_K B) \cong (L \otimes_K A) \otimes_L (L \otimes_K B)$ as L-algebras, res is a ring homomorphism.

(b) To see that $\text{res}_{K \to L}(\ker \beta_K) \subseteq \ker \beta_L$ note that if $n \in \mathbf{Z}^+$, then

$$\text{res}_{K \to L}([M_n(K)]_K - [K]_K) = [M_n(L)]_L - [L]_L.$$

The inclusion thus holds by 1.5, and the fact that res is a ring homomorphism. ■

The following lemma shows how the functorial properties of A_{AZ} and S coincide. A similar result can be proven for A_{Br} and B.

5.5 LEMMA. *Let E/D be a finite Galois extension, with Galois group G. Let $H \leq G$ and set $L = E^H$. Let η: $G/H \to G/G$ be the canonical map. Then the following diagrams commute.*

$$\begin{array}{ccc} A_{\mathrm{AZ}}(H) & \xrightarrow{\eta^*} & A_{\mathrm{AZ}}(G) \\ {\scriptstyle\Psi_H}\downarrow & & \downarrow{\scriptstyle\Psi_G}, \\ S(E,L) & \xrightarrow[\mathrm{ind}_{L\to}]{} & S(E,D) \end{array} \qquad \begin{array}{ccc} A_{\mathrm{AZ}}(G) & \xrightarrow{\eta^*} & A_{\mathrm{AZ}}(H) \\ {\scriptstyle\Psi_G}\downarrow & & \downarrow{\scriptstyle\Psi_H}. \\ S(E,D) & \xrightarrow[\mathrm{res}_{D\to L}]{} & S(E,L) \end{array}$$

where Ψ_H, Ψ_G are the isomorphisms of 4.6.

Proof. Let $[S, \alpha, (A)] \in A_{\mathrm{AZ}}(H)$. Then

$$\Psi_G\eta^*([S,\alpha,(A)]) = \Psi_G([S,(A)]) = [A]_D = [A_\alpha]_D = \mathrm{ind}_{L\to D}([A_\alpha]_L) \\ = \mathrm{ind}_{L\to D}\Psi_H([S,\alpha,(A)]).$$

Let $[S,(A)] \in A_{\mathrm{AZ}}(G)$. Let π_H and π_S be the projections of $G/H \times S$. By 4.4, $\pi_S^0(A) = (R_{G/H\times S} \otimes_{R_S} A) = (R_{G/H} \otimes_{R_{G/G}} A) = (R_{G/G} \otimes_D A)$ (identifying $R_{G/G}$ with D). Moreover, π_{H*}: $R_{G/H} \to R_{G/H\times S} \cong R_{G/H} \otimes_D R_S$ is injection: $\pi_{H*}(x) = x \otimes 1$. Thus the identity map defines an $R_{G/H}$-algebra isomorphism

$$R_{G/H} \otimes_D A \to \left(R_{G/H} \otimes_D A\right)_{\pi_H}.$$

Therefore,

$$\Psi_H\eta_*([S,(A)]) = \Psi_H([G/H \times S, \pi_H, \pi_S^0(A)]) = \left[\left(R_{G/H} \otimes_D A\right)_{\pi_H}\right]_L \\ = [L \otimes_D A]_L = \mathrm{res}_{D\to L}([A]_F) = \mathrm{res}_{D\to L}\Psi_G([S,(A)]). \quad \blacksquare$$

We are interested in looking at ker($\mathrm{res}_{D\to L}$) and im($\mathrm{ind}_{L\to D}$); it will be convenient to proceed more generally. Let M be any Green-functor: $\mathcal{G} \to$ AB, and let S be a G-set. Denote by $K_M(S)$ the kernel of the map $M_*(\eta_S)$: $M(G) = M(G/G) \to M(S)$, and by $I_M(S)$ the image of $M^*(\eta_S)$: $M(S) \to M(G)$.

5.6 PROPOSITION. *Let G be a finite group, and M: $\mathcal{G} \to$ AB a Green-functor. Then for any G-set S,*

(a) $|G| \cdot (I_M(S) \cap K_M(S)) = 0$,

(b) $|G| \cdot M(G) \subseteq I_M(S) + K_M(S)$.

Proof. The condition that M be a Green-functor is stronger than necessary. For a proof, see [5].

Combining 5.2, 5.5 and 5.6, we obtain:

5.7 COROLLARY. *Let E/D be a finite Galois extension with Galois group G. Let $H \leq G$, and set $L = E^H$. Then*
(a) $\operatorname{im}(\operatorname{ind}_{L \to D}) \cap \ker(\operatorname{res}_{D \to L}) = 0$,
(b) $|G| \cdot S(E, D) \subseteq \operatorname{im}(\operatorname{ind}_{L \to D}) + \ker(\operatorname{res}_{D \to L})$.

What this corollary says about associative algebras is expressed by the following two theorems.

5.8 THEOREM. *Let L/D be a finite separable field extension, and let A, B be separable L-algebras. If $L \otimes_D A \cong L \otimes_D B$ as L-algebras, then $A \cong B$ as D-algebras.*

Proof. By a standard characterization of separable algebras over fields, A and B may be expressed as finite products of finite dimensional, simple L-algebras, where each simple algebra has as center a finite separable field extension of L (see [15]). Since L/D is finite separable, it follows that there is a finite Galois extension E/D containing the centers of all of these simple algebras. Thus, $A, B \in \mathrm{SEP}(E, L)$. Consider $[A]_D - [B]_D \in S(E, D)$. Plainly

$$\operatorname{ind}_{L \to D}([A]_L - [B]_L) = [A]_D - [B]_D.$$

Also,

$$\operatorname{res}_{D \to L}([A]_D - [B]_D) = [L \otimes_D A]_L - [L \otimes_D B]_L = 0,$$

since $L \otimes_D A \cong L \otimes_D B$ as L-algebras. Thus

$$[A]_D - [B]_D \in \operatorname{im}(\operatorname{ind}_{L \to D}) \cap \ker(\operatorname{res}_{D \to L}) = 0,$$

so that $[A]_D = [B]_D$. By 1.2(c), $A \cong B$ as D-algebras. ■

5.9 THEOREM. *Let E/D be a finite Galois extension, and suppose $A \in$ $\mathrm{SEP}(E, D)$. Then there are algebras $D, C \in \mathrm{SEP}(E, D)$ with $E \otimes_D B \cong E \otimes_D C$ as E-algebras, and there are algebras $Y, Z \in \mathrm{SEP}(E, E)$ (that is, finite products of central simple E-algebras), such that $A \dotplus \cdots \dotplus A \dotplus B \dotplus Y \cong C \dotplus Z$ as D-algebras, where $[E : D]$ copies of A appear in the left hand product.*

Proof. Take $H = \{1\}$ in 5.7(b), so that $L = E$. Set $n = [E : D] = |Gal(E/D)|$. Then by 5.7(b),

$$n[A]_D = \operatorname{ind}_{E \to D}([Z]_E - [Y]_E) + ([C]_D - [B]_D),$$

where $[C]_D - [B]_D \in \ker(\operatorname{res}_{D \to E})$. Thus, $n[A]_D + [Y]_D + [B]_D = [Z]_D + [C]_D$, and the result follows from 1.2(c). ■

The purpose of the remainder of this section is to describe the functorial properties of the Green-functor A_F more fully. Although these results tell us nothing new about the Brauer ring, they do underscore the reasons for calling this particular Brauer ring the best possible choice.

For the remainder of this section, fix a finite group G. We shall denote by AM^G the category of additive contravariant functors F: $\mathscr{G} \to \mathrm{AM}$, with natural transformations as morphisms, and by GF^G the category of Green-functors M: $\mathscr{G} \to \mathrm{AB}$, with binatural transformations as morphisms. Given $M \in \mathrm{GF}^G$, it follows from 5.1(b) that $M_* \in \mathrm{AM}^G$, where for $S \in \mathscr{G}$ we have $M_*(S) = M(S)$. We thus obtain the forgetful functor U: $\mathrm{GF}^G \to \mathrm{AM}^G$ given by $U(M) = M_*$. The point of our present discussion is to show that the correspondence $F \to A_F$ from AM^G to FG^G is the left adjoint to the functor U. We must first show how this correspondence defines a functor. Let $F_1, F_2 \in \mathrm{AM}^G$, and let γ: $F_1 \to F_2$ be a natural transformation. Then there is an induced natural transformation of Green-functors $\hat{\gamma}$: $A_{F_1} \to A_{F_2}$, such that for all $S \in G$, $[T, \phi, x] \in A_{F_1}(S)$, we have

$$\hat{\gamma}_S([T, \phi, x]) = [T, \phi, \gamma_T(x)] \in A_{F_2}(S).$$

5.10 Proposition. *The correspondences $F \to A_F$, $\gamma \to \hat{\gamma}$ define a covariant functor from AM^G to GF^G.*

Proof. There are many minor points to check, we leave them all to the reader. ∎

Let $F \in \mathrm{AM}^G$, and $M \in \mathrm{GF}^G$, and let ρ: $F \to U(M)$ be a natural transformation. Define, for $S \in G$, a map

$$\tilde{\rho}_S\colon A_F(S) \to M(S) \text{ by } \tilde{\rho}_S([T, \phi, x]) = M^*(\phi)(\rho_T(x)),$$

all $[T, \phi, x] \in A_F(S)$. Then, as is readily seen, $\tilde{\rho} = \{\tilde{\rho}_S\colon S \in \mathscr{G}\}$ defines a natural transformation of Green-functors $A_F \to M$.

5.11 Theorem. *The functor $F \to A_F$ from* AM^G *to* GF^G *is the left adjoint of the forgetful functor U:* $\mathrm{GF}^G \to \mathrm{AM}^G$.

Proof. Let $F \in \mathrm{AM}^G$, $M \in \mathrm{GF}^G$. The natural bijection

$$Mor(A_F, M) \leftrightarrow Mor(F, U(M))$$

is then given by inverse bijections described as follows. Define

$$\Phi\colon Mor(A_F, M) \to Mor(F, U(M))$$

by $\Phi(\gamma)_S(x) = \gamma_S([S, 1_S, x])$, all $S \in G$, $x \in F(S)$; and

$$\Psi\colon Mor(F, U(M)) \to Mor(A_F, M)$$

by $\Psi(\rho) = \tilde{\rho}$. ∎

If we let $M = A_F$, then adjointness implies that the identity transformation $1_{A_F} \in \mathrm{Mor}(A_F, A_F)$ determines a universal arrow $\Phi(1_{A_F})\colon F \to U(A_F)$. Explicitly, we have $\Phi(1_{A_F})_S(x) = [S, 1_S, x]$, all $S \in \mathcal{G}$, $x \in F(S)$, and the universality may be rephrased thus:

5.12 COROLLARY. *Given any Green-functor M, and natural transformation $\rho\colon F \to U(M)$, there is a natural transformation of Green-functors $\tilde{\rho}\colon A_F \to M$ such that $\rho = \tilde{\rho}\Phi(1_{A_F})$.*

The trivial functor $I\colon \mathcal{G} \to \mathrm{AM}$ is both an initial and final object in AM^G. For each $F \in \mathrm{AM}^G$, let $\alpha_F\colon I \to F$, and $\xi_F\colon F \to I$ be the canonical natural transformations. Since $\xi_F \alpha_F$ is the identity map, it follows that for each G-set S, $\hat{\alpha}_{F,S}\colon A_I(S) \to A_F(S)$ embeds $A_I(S)$ as a direct summand of $A_F(S)$, and that $\hat{\alpha}_{F,S}\colon A_F(S) \to A_I(S)$ is surjective. Using the fact that $\hat{\alpha}_F$ and $\hat{\xi}_F$ are natural transformations of Green-functors, together with the fact that $A \cong A_I$ is an initial object in GF^G, we obtain our final result.

5.13 COROLLARY. *For any $F \in \mathrm{AM}^G$, A_F is an initial object in the category of Green-functors* GF^G.

6. Comments

6.1. We have done little to relate the structure of the Brauer ring $B(F)$ to the properties of the field F. Of particular interest of $B(\mathbf{Q})$ and $B(F_{ab}, F)$, where F_{ab} is the maximal abelian unramified extension of the number field F. It may also be interesting to look at the case when E is the maximal abelian field extension of a field F which contains n-th roots of unity for all n. To begin with, it would be valuable to have information on the subalgebra of $B(E, F)$ in which the division algebras are restricted to those with n as exponent (fixed n), where E is the maximal abelian extension with exponent n and F has a primitive n-th root of unity.

6.2. The category theory we've used is only a fraction of what's available; in particular, 5.6 can be strengthened considerably. The reader who is interested in these aspects may profit by consulting [5] or [12].

6.3. There are homology and cohomology theories available for the Green-functors A_F (see [5]). It may be interesting to see how the cohomology groups for the functor A_{Br}, and the Galois cohomology groups are related.

6.4. By analogy with Dress' work on the prime ideals of the Burnside ring [4], for a certain class of additive contravariant functors the prime ideal structure of $A_F(G)$ can be described. In particular, the prime ideals of $B(E, \mathbf{Q}_p)$ ($E/\mathbf{Q}_p$ finite Galois) are known. For details, consult Chapter 6 of [8].

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