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On the Adèle rings of radical extensions of the rationals

By

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1. Introduction. In 1926, Gassman [2] discovered that algebraic number fields are not completely determined by their zeta functions. Subsequently, Perlis [9] produced two infinite families of pairs of nonisomorphic fields with identical zeta functions. Following Perlis, we call two number fields *arithmetically equivalent* if their zeta functions coincide.

Komatsu [7, 8] showed that if $k \geqq 3$ and $X^{2^k} - a$ is irreducible over $\mathbb{Q}$, then the fields $\mathbb{Q}(\sqrt[2^k]{a})$ and $\mathbb{Q}(\sqrt[2^k]{a} \cdot \sqrt{2})$ are arithmetically equivalent. Under the added assumption that a is square-free, he also settled the more delicate question of determining when the Adèle rings of these fields are isomorphic.

Our present effort amounts to a complete description of those integers n, a, b for which the Adèle rings of $\mathbb{Q}(\sqrt[n]{a})$ and $\mathbb{Q}(\sqrt[n]{b})$ are isomorphic. Along the way we will also give a characterization of when these two fields are arithmetically equivalent. Of course, if these fields are isomorphic, then there is nothing to be done, so it is important to be able to recognize such an isomorphism. This problem, however, has already been solved (see [1, 10]). In order to state the result, we develop some terminology.

In what follows we shall have no need for explicit mention of the zeta function (see 1.2), thus the notation ζ_n is reserved for a primitive n-th root of unity. For any t, set $\eta_t = \zeta_{2^t} + \zeta_{2^t}^{-1}$. For any field F (with $\mathrm{char}(F) \neq 2$) define T as follows: if $\eta_t \in F$ for all t, set $T = \infty$, otherwise let $T = \max\{t\colon \eta_t \in F\}$. This is well defined since if $\eta_t \in F$ then the equality $\eta_t^2 = 2 + \eta_{t-1}$ implies that $\eta_j \in F$ for all $j \leqq t$.

1.1 Theorem. *Let F be a field with* $\mathrm{char}(F) \neq 2$ *and suppose $X^n - a$, $X^n - b$ are irreducible over F. Then the fields $F(\sqrt[n]{a})$, $F(\sqrt[n]{b})$ are F-isomorphic if and only if one of the following holds*:

(i) $a b^i \in F^n$ *for some i with* $(i, n) = 1$, *or*
(ii) $T < \infty$, $2^{T+1} \mid n$, $-a \in F^2$, $-b \in F^2$, *and* $a b^i (2 + \eta_T)^{n/2} \in F^n$ *for some i with* $(i, n) = 1$.

Remark. Since $\zeta_4 + \zeta_4^{-1} = 0$ and $\zeta_8 + \zeta_8^{-1} = \sqrt{2}$, it follows that in the case $F = \mathbb{Q}$ we have $T = 2$. Note the similarity of 1.1 with 1.3 to follow.

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If F is an algebraic number field, then $\mathcal{O}_F$ denotes the ring of integers in F, while $\mathscr{A}_F$ will denote the Adèle ring of F. For any prime P of F, F_P is the completion of F at P. If p is a rational prime, and p factors $p\mathcal{O}_F = P_1^{e_1} \cdot \ldots \cdot P_g^{e_g}$, where the inertial degrees $f_i = f(P_i/p)$ are arranged so that $f_i \leqq f_{i+1}$, then the tuple $S_F(p) = (f_1, \ldots, f_g)$ is called the *splitting type* of p in F. The importance of splitting types to our work is indicated by the following result ([9], Theorem 1).

1.2 Theorem. *Let E and F be two algebraic number fields. Then the following are equivalent.*

a) *E and F are arithmetically equivalent.*
b) *$S_E(p) = S_F(p)$ for all but finitely many primes $p \in \mathbb{Z}$.*
c) *$S_E(p) = S_F(p)$ for all primes $p \in \mathbb{Z}$.*

Furthermore, if E and F are arithmetically equivalent, then their Galois closures coincide.

If K/F is a finite extension of number fields, we denote $S(K/F) = \{P \subset \mathcal{O}_F$: P has a prime divisor in $\mathcal{O}_K$ of relative inertial degree 1 over $P\}$. Two fields K_1, K_2 are said to be *Kronecker equivalent* over F if the sets $S(K_1/F)$, $S(K_2/F)$ differ by at most finitely many elements. We will need the main result of Gerst [3].

1.3 Theorem. *Let $X^n - a$, $X^n - b$ be irreducible over $\mathbb{Q}$. Then the fields $\mathbb{Q}(\sqrt[n]{a})$, $\mathbb{Q}(\sqrt[n]{b})$ are Kronecker equivalent over $\mathbb{Q}$ if and only if either*

(i) *$ab^i \in \mathbb{Q}^n$ for some i with $(i, n) = 1$, or*
(ii) *$8 \mid n$ and $ab^i 2^{n/2} \in \mathbb{Q}^n$ for some i with $(i, n) = 1$.*

By 1.2, note that if K_1, K_2 are arithmetically equivalent then they are Kronecker equivalent over $\mathbb{Q}$. The reverse implication does not hold since Kronecker equivalence does not even imply that the fields have the same degree over $\mathbb{Q}$. Even if one restricts to fields of the same degree, Kronecker equivalence does not imply arithmetic equivalence for the following reason. If K_2 is arithmetically equivalent to K_1, then K_2 is contained in the Galois closure of K_1, so there are only finitely many fields arithmetically equivalent to K_1. However, Jehne [5] has shown that if $L/\mathbb{Q}$ is cyclic of odd order, then there are infinitely many fields K with $[K : L] = 2$ with each K Kronecker equivalent to L (and thus to each other). Thus, if K is any one of these, then there are infinitely many fields Kronecker equivalent to K having the same degree over $\mathbb{Q}$. Thus, there must exist a pair of such fields which are Kronecker equivalent but not arithmetically equivalent.

Finally, for the study of Adèle rings, we state the following result due to Iwasawa ([4], Lemma 7).

1.4 Theorem. *If K_1, K_2 are two algebraic number fields, then the Adèle rings of K_1 and K_2 are isomorphic if and only if for every prime $p \in \mathbb{Z}$ there is a bijection between the prime divisors of p in K_1 with the prime divisors of p in K_2 such that the corresponding completions are isomorphic. That is, $\mathscr{A}_{K_1} \cong \mathscr{A}_{K_2}$ if and only if $K_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong K_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p$, for all primes $p \in \mathbb{Z}$.*

1.5 Corollary. *If K_1, K_2 are two number fields with isomorphic Adèle rings, then K_1, K_2 are arithmetically equivalent.*