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The Burnside Ring Modulo a Prime

ELIOT JACOBSON

Department of Mathematics, Ohio University, Athens, Ohio 45701

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1. INTRODUCTION

Let G be a finite group, and let $B(G)$ denote the Burnside ring of G, that is, the Grothendieck ring constructed on the category of finite G-sets under disjoint union and cartesian product (see [2] for a more complete description). As Solomon [9] pointed out, the structure of $B(G)$ becomes trivial upon scalar extension to $\mathbb{Q}$; indeed, $\mathbb{Q} \otimes_{\mathbb{Z}} B(G)$ is isomorphic with a finite product of copies of $\mathbb{Q}$. It is natural to ask for a similar result when $\mathbb{Q}$ is replaced by the prime field $\mathbb{Z}_p$ of integers modulo p. The purpose of the present paper is to determine the structure of the ring $B_p(G) = \mathbb{Z}_p \otimes_{\mathbb{Z}} B(G)$. As we shall see, if $p \mid |G|$, then the structure of $B_p(G)$ is quite complex.

We begin with some notational conventions. Let $P = P(G)$ denote the set of all conjugacy classes of subgroups of G. For each $b \in P$, pick a representative H_b of b, and let S_b denote the transitive G-set of cosets modulo H_b. Let $\imath$ denote the class of the trivial subgroup, so that $S_\imath = G/\{1\}$. The set P has a natural partial ordering, where we write $a \leqslant b$ whenever H_a is subconjugate to H_b (denoted $H_a \lesssim H_b$). Then $\imath$ is the unique minimal element of P. For $a, b, c \in P$, let V_{abc} be the number of orbits in $S_a \times S_b$, under the diagonal action of G, which are isomorphic with S_c as G-sets. Equivalently, write $G = \bigcup_{i=1}^{r} H_a \sigma_i H_b$, then $V_{abc} = |\{i\colon H_a \cap {}^{\sigma_i}H_b \sim H_c\}|$, where $\sim$ denotes G-conjugacy. For brevity, denote $V_{abb} = V_{ab}$ and $V_{bbb} = V_b$. For a G-set S, we let $[S]$ denote its image in $B(G)$. The following proposition collects some well-known properties of $B(G)$ (see [3]).

1.1. PROPOSITION. (a) *Additively, $B(G)$ is free on the set $\{[S_a]\colon a \in P\}$, that is, $\{S_a\colon a \in P\}$ is a complete set of representatives of isomorphism classes of transitive G-sets.*

(b) *For any $a, b \in P$, $[S_a][S_b] = \sum_{c \in P} V_{abc}[S_c]$, that is, the V_{abc} are structure constants for $B(G)$.*

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(c) *For any* $a, b, c \in P$, $V_{abc} = 0$ *unless both* $c \leqslant a$ *and* $c \leqslant b$.

(d) *For any* $a \in P$, $V_a = (N_G(H_a) : H_a)$.

For the remainder of this paper, assume we have a fixed finite group G. For brevity, we denote $B_p = B_p(G)$, and $B = B(G)$. Even though B does not embed faithfully in B_p, the natural isomorphism $B_p \simeq B/pB$ implies that we may consider $B_p = \bigoplus_{a \in P} \mathbb{Z}_p[S_a]$, where multiplication is given by 1.1(b) by reducing the V_{abc} modulo p.

2. The Radical of B_p

If S is a G-set, and $H \leqslant G$, then S^H denotes the set of H fixed points in S. For any $a \in P$, there is a ring homomorphism $\phi_a : B \to \mathbb{Z}$ given by $\phi_a([S]) = |S^{H_a}|$ (see [2]), in particular, $\phi_a([S_b]) = V_{ba}$. The product map $\phi = (\phi_a): B \to \prod_{a \in P} \mathbb{Z}$ is injective. By reducing modulo p, there are induced ring homomorphisms $\psi_a : B_p \to \mathbb{Z}_p$ and $\psi = (\psi_a): B_p \to \prod_{a \in P} \mathbb{Z}_p$, where $\psi_a([S_b]) = V_{ba} \in \mathbb{Z}_p$. However, ψ need not be injective.

2.1. Lemma. *ψ is injective if and only if $p \nmid |G|$. In this case, ψ is an isomorphism.*

Proof. $\Rightarrow$ If $p \mid |G|$, then $\psi_\iota([S_\iota]) = V_{\iota\iota} = |G| = 0$, and for any other $a \in P$, $\psi_a([S_\iota]) = V_{\iota a} = 0$ by 1.1(c), since $a \nleqslant \iota$. Thus, $[S_\iota] \in \ker \psi$.

$\Leftarrow$ Suppose $0 \neq x = \sum_b r_b [S_b] \in \ker \psi$, and choose $g \in P$ maximal with $r_g \neq 0$. For any other $c \in P$, $r_c \neq 0$ implies $g \nleqslant c$, so that $V_{cg} = 0$. Then, $0 = \psi_g(x) = \sum_b r_b V_{bg} = r_g V_g$, so that $p \mid V_g$. It follows that $p \mid |G|$. By counting $\mathbb{Z}_p$-dimensions, ψ is injective if and only if it is an isomorphism. ∎

We let $J_p = J(B_p)$ denote the Jacobson radical of B_p.

2.2. Proposition. $J_p = \ker \psi$.

Proof. $\subseteq$ Recall that J_p is the intersection of the annihilators of all simple B_p-modules. For any $a \in P$, $\mathbb{Z}_p$ becomes a simple B_p-module via ψ_a. Thus, if $x \in J_p$, then $\psi_a(x) \cdot \mathbb{Z}_p = 0$, so that $\psi_a(x) = 0$, all $a \in P$. It follows that $x \in \ker \psi$.

$\supseteq$ If $p \nmid |G|$, then $\ker \psi = 0$ by 2.1, and the inclusion is clear. Thus we may assume $p \mid |G|$. Let $0 \neq x = \sum_b r_b [S_b] \in \ker \psi$. It is sufficient to show that x is nilpotent, and this we accomplish by induction on $n(x) = \max\{|H_b| : r_b \neq 0\}$. If $n(x) = 1$, then $x = r_\iota [S_\iota]$, so that $x^2 = r_\iota^2 |G| [S_\iota] = 0$ since $p \mid |G|$. Assume $n(x) > 1$. It is enough to show that $n(x^2) < n(x)$, since induction then implies x^2 (hence x also) is nilpotent. Since $x \in \ker \psi$, we have the equations:

$$\sum_b r_b V_{ba} = 0 \qquad \text{all } a \in P. \tag{1}$$

Therefore,

$$\begin{aligned} x^2 &= \sum_{b,c,d} r_b r_c V_{bcd}[S_d] \\ &= \sum_c r_c \left(\sum_b r_b V_{bc} \right) [S_c] + \sum_{\substack{b,c \\ d<c}} r_b r_c V_{bcd}[S_d] \\ &= \sum_{\substack{b,c \\ d<c}} r_b r_c V_{bcd}[S_d] \qquad \text{(by (1))}; \end{aligned}$$

thus

$$x^2 = \sum_d \left(\sum_{c>d} r_c \left(\sum_b r_b V_{bcd} \right) \right) [S_d] = \sum_d t_d [S_d]. \tag{2}$$

Fix $g \in P$ with $r_g \neq 0$ and $n(x) = |H_g|$. It is sufficient to show that $t_d \neq 0$ implies $|H_d| < |H_g|$. By (2), if $t_d \neq 0$, then there exists $c > d$ with $r_c \neq 0$. Then $|H_d| < |H_c| \leqslant |H_g|$. ▮

2.3. Corollary. *B_p is semisimple if and only if $P \nmid |G|$. In fact, if $p \nmid |G|$, then $B_p \simeq \mathbb{Z}_p \dotplus \cdots \dotplus \mathbb{Z}_p$, $|P|$-times.*

3. The Dimension of the Radical

Fix any $a, b \in P$, and assume that $a \leqslant b$. Define $T = \{\sigma \in G \colon H_a^\sigma \subseteq H_b\}$. If $\sigma \in T$, then $H_a \sigma N_G(H_b) \subseteq T$, so that T is a union of $H_a - N_G(H_b)$ double cosets. Fix a double coset decomposition $T = \dot{\bigcup}_{i=1}^k H_a \tau_i N_G(H_b)$ (possibly $k = 0$), and write $G = T \mathbin{\dot\cup} \dot{\bigcup}_{i=1}^s H_a \sigma_i H_b$, where $H_a^{\sigma_i} \nsubseteq H_b$ for $1 \leqslant i \leqslant s$. Let $t = V_b = (N_G(H_b) \colon H_b)$, and write $N_G(H_b) = \rho_1 H_b \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \rho_t H_b$.

3.1. Lemma. *If $H_a \tau_i \rho_j H_b = H_a \tau_i \rho_k H_b$, then $j = k$.*

Proof. Write $\tau_i \rho_j = h_1 \tau_i \rho_k h_2$, for some $h_1 \in H_a$, $h_2 \in H_b$. Then $\rho_k^{-1} \rho_j = \rho_k^{-1} \tau_i^{-1} \tau_i \rho_j = \rho_k^{-1} \tau_i^{-1} h_1 \tau_i \rho_k h_2 \in H_b$, since $\tau_i^{-1} h_1 \tau_i \in H_b$, and $\rho_k \in N_G(H_b)$. Thus $\rho_j H_b = \rho_k H_b$, so that $j = k$. ▮

3.2. Proposition. *For any $a, b \in P$, $V_b \mid V_{ba}$.*

Proof. If $a \nleq b$, then $V_{ba} = 0$ and the result is clear. Assume $a \leqslant b$. Then by 3.1, we have (with the above notation) the *disjoint* double coset decomposition:

$$G = \dot{\bigcup_{i=1}^{k}} \dot{\bigcup}_{j=1}^{t} H_a \tau_i \rho_j H_b \cup \dot{\bigcup}_{i=1}^{s} H_a \sigma_i H_b.$$

Therefore,

$$\begin{aligned} S_a \times S_b &= G/H_a \times G/H_b \\ &\simeq \dot{\bigcup}_{i=1}^{k} \dot{\bigcup}_{j=1}^{t} G/H_b \cap H_a^{\tau_i \rho_j} \cup \dot{\bigcup}_{i=1}^{s} G/H_b \cap H_a^{\sigma_i} \\ &\simeq \dot{\bigcup}_{i=1}^{k} \dot{\bigcup}_{j=1}^{t} G/H_a \cup \dot{\bigcup_{i=1}^{s}} G/H_{a_i} \qquad \text{where} \quad a_i < a, \text{ all } i \\ &= (kt) \cdot S_a \cup \dot{\bigcup_{i=1}^{s}} S_{a_i}. \end{aligned}$$

We conclude that $V_{ba} = kt = kV_b$, so $V_b \mid V_{ba}$. ∎

For the remainder of this paper, "dimension" will mean dimension as a $\mathbb{Z}_p$-space.

3.3. Proposition. $\dim(J_p) = |\{a : p \mid V_a\}|$.

Proof. If $p \nmid |G|$ the result is clear, so assume that $p \mid |G|$. Extend the partial ordering $\leqslant$ on P to a total ordering so that $H_a \lesssim H_b$ implies $a \leqslant b$ (but not necessarily the converse). Set $n = |P|$, and define a map $B_p \to \mathbb{Z}_p^{(n)} = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (n-times) by sending $x = \sum_b r_b [S_b]$ to the vector $\bar{x} = (..., r_b, ...)$, ordered by $\leqslant$. Then $x \to \bar{x}$ defines a $\mathbb{Z}_p$-space isomorphism. Let M be the matrix (V_{ba}), considered as a linear transformation by right multiplication on $\mathbb{Z}_p^{(n)}$. Note that $x = \sum_b r_b [S_b] \in J_p$ iff $x \in \ker \psi$ iff $\sum_b r_b V_{ba} = 0$ all $a \in P$ iff $\bar{x} \cdot M = 0$ iff $\bar{x} \in \ker M$. Hence the map $x \to \bar{x}$ restricts to a $\mathbb{Z}_p$-space isomorphism $J_p \to \ker M$. In particular, we conclude that $\dim(J_p) + \operatorname{rank}(M) = |P|$. Since M *is* lower triangular, and since a zero diagonal entry occurs if and only if the entire row is zero (by 3.2), it follows that $\operatorname{rank}(M) = |\{a : p \nmid V_a\}|$, giving our result. ∎

4. Idempotents in B_p

In this section we use the map ψ to construct the primitive idempotents of B_p. Fix a prime $p \in \mathbb{Z}$, then define $P^* = P^*(G) = \{a : p \nmid V_a\}$. Let $n(p) = |P^*|$. By 3.3, $B_p/J_p \simeq \operatorname{im}(\psi)$ is a semisimple subalgebra of $\Pi_{a \in P} \mathbb{Z}_p$

of dimension $n(p)$. By Wedderburn's theorem, $\mathrm{im}(\psi)$ is isomorphic with an algebra product of field extensions of $\mathbb{Z}_p$. By noting that no element in $\mathrm{im}(\psi)$ has multiplicative order larger than $p-1$, it follows that there is an algebra isomorphism $\sigma\colon \mathrm{im}(\psi) \to \prod_{i=1}^{n(p)} \mathbb{Z}_p$. Let $\delta_i \in \prod_{i=1}^{n(p)} \mathbb{Z}_p$ be the primitive idempotent corresponding to projection onto the ith coordinate. For each i, choose $x_i \in B_p$ with $\sigma\psi(x_i) = \delta_i$. Then $\sigma\psi(x_i^2 - x_i) = 0$, so that $x_i^2 - x_i \in \ker \psi = J_p$, that is, x_i is idempotent modulo J_p. By a standard lifting theorem [7, p. 97], there are idempotents $e_i \in B_p$ such that $e_i \equiv x_i$ (mod J_p), $i = 1,..., n(p)$. We now establish some properties of the idempotents $\{e_i : i = 1,..., n(p)\}$.

4.1. Lemma. *The idempotents* $\{e_i : i = 1,..., n(p)\}$ *are distinct, pairwise orthogonal, and sum to* 1. *Moreover, each* e_i *is primitive.*

Proof. If $e_i = e_j$ then $\delta_i = \psi(e_i) = \psi(e_j) = \delta_j$, so $i = j$. For orthogonality, if $i \neq j$ then $\sigma\psi(e_i e_j) = \delta_i \delta_j = 0$, hence $e_i e_j \in \ker \psi = J_p$. Since $e_i e_j$ is idempotent, and J_p is nilpotent, $e_i e_j = 0$. To see that $\sum_i e_i = 1$, note that $\sigma\psi(\sum_i e_i) = \sum_i \delta_i = 1 = \sigma\psi(1)$, so that $1 - \sum_i e_i \in J_p$. As above, this implies $1 - \sum_i e_i = 0$.

To see that each e_i is primitive, let $f \in B_p$ be idempotent with $fe_i \neq 0$. We must show $fe_i = e_i$. Write $\sigma\psi(f) = \sum_{j \in S} \delta_j$, for some $S \subseteq \{1,..., n(p)\}$. If $i \notin S$ then $0 = \sigma\psi(f)\,\delta_i = \sigma\psi(fe_i)$ implies $fe_i \in J_p$, so that $fe_i = 0$ contrary to assumption. Hence $i \in S$. Since $f \equiv \sum_{j \in S} e_j$ (mod J_p), we have $fe_i \equiv e_i$ (mod J_p). As in the preceding paragraph, this implies $fe_i = e_i$. ▮

To aid in the statement of the next lemma, we (temporarily) write $e_i = \sum_a c_{ia}[S_a]$, some $c_{ia} \in \mathbb{Z}_p$, $i = 1,..., n(p)$.

4.2. Lemma. (a) *For any* i, *if* $a \in P$ *is maximal with* $c_{ia} \neq 0$, *then* $a \in P^*$ *and* $c_{ia} = V_a^{-1}$.

(b) *Suppose* $i \neq j$, *and choose* a, b *maximal with* c_{ia}, $c_{jb} \neq 0$, *respectively. Then* $a \neq b$.

(c) *Each idempotent* e_i *contains a unique maximal* $a \in P^*$ *with* $c_{ia} \neq 0$. *If* $i \neq j$, *then these maximal elements are distinct.*

Proof. A simple computation using $e_i^2 = e_i$ and maximality of a yields $e_i \equiv c_{ia}^2 V_a[S_a]$ (mod $\oplus_{b \neq a} \mathbb{Z}_p[S_b]$). Thus $c_{ia} = c_{ia}^2 V_a \neq 0$, so $a \in P^*$ and $c_{ia} = V_a^{-1}$, proving (a).

(b) If $a = b$, then $0 = e_i e_j \equiv c_{ia} c_{ja} [S_a]^2$ (mod $\oplus_{b \neq a} \mathbb{Z}_p[S_b]) \equiv c_{ia} c_{ja} V_a [S_a]$ (mod $\oplus_{b \neq a} \mathbb{Z}_p[S_b]$). This implies $c_{ia} c_{ja} V_a = 0$, contrary to (a).

(c) Let n_i be the number of maximal elements occuring in e_i. Then by (b), $|P^*| \geqslant \sum_i n_i \geqslant \sum_i 1 = |P^*|$, forcing $n_i = 1$, all i. The second statement restates part (b). ▮

For each $a \in P^*$ let $e_a = \sum_b \lambda_{ab}[S_b]$ be the primitive idempotent of B_p with $\lambda_{aa} = V_a^{-1}$ and $\lambda_{ab} = 0$ if $b \nleq a$. By 4.2 the idempotents $\{e_a : a \in P^*\}$ are just the idempotents $\{e_i : i = 1,..., n(p)\}$ relabeled. Relabel the x_i's and δ_i's accordingly, so that $\sigma\psi(x_a) = \delta_a$ and $e_a \equiv x_a \pmod{J_p}$. For $b \in P^*$, let $\pi_b : \prod_{a \in P^*} \mathbb{Z}_p \to \mathbb{Z}_p$ be projection onto the b-th coordinate.

4.3. THEOREM. *Let G be a finite group, and $p \in \mathbb{Z}$ be a prime. Then $B_p(G) \simeq \prod_{a \in P^*} L_a$, where for $b \in P^*$, $L_b = B_p(G)\, e_b$ is a local ring with residue field $\mathbb{Z}_p$.*

Proof. For each $b \in P^*$, define $\tau_b : B_p e_b \to \mathbb{Z}_p$ by $\tau_b = \pi_b \sigma\psi$. We claim that $J(B_p e_b) = \ker \tau_b$. To see this suppose first that $xe_b \in J(B_p e_b)$. Then $xe_b \in B_p$ is nilpotent, so that $xe_b \in J(B_p) = \ker \psi \subseteq \ker \tau_b$. Conversely, let $xe_b \in \ker \tau_b$. Note that $\sigma\psi(xe_b) = \sigma\psi(xe_b e_b) = \sigma\psi(xe_b)\, \delta_b$. Since $\pi_b : (\prod_{a \in P^*} \mathbb{Z}_p)\, \delta_b \to \mathbb{Z}_p$ is an isomorphism, it follows that $\sigma\psi(xe_b) = 0$, that is, $xe_b \in \ker \psi = J(B_p)$. Then $xe_b \in J(B_p)\, e_b = J(B_p e_b)$, finishing the claim. By the first homomorphism theorem we conclude $B_p e_b / J(B_p e_b) \simeq \mathbb{Z}_p$, so $B_p e_b$ is a local ring with residue field $\mathbb{Z}_p$. The remainder of the theorem is clear, since by 4.1, $B_p = B_p(\sum_a e_a) \simeq \prod_a (B_p e_a)$. ▮

4.4. COROLLARY. *G is a p-group if and only if $B_p(G)$ is a local ring.*

Proof. $\Rightarrow$ If G is a p-group and $H \subset G$, then $H \subset N_G(H)$. Thus, $|P^*| = 1$, and the result follows from 4.3.

$\Leftarrow$ Let H be p-Sylow in G. Then $p \nmid (N_G(H) : H)$, so H represents a class in P^*. However, if $B_p(G)$ is local, then by 4.3, $|P^*| = 1$. Thus H and G are conjugate, that is, $H = G$. ▮

5. DIMENSION OF THE LOCAL DIRECT FACTORS

Fixed throughout this section is a prime $p \in \mathbb{Z}$. For any $a \in P^*$ and $b \in P$, $\psi_b(e_a)$ is an idempotent in $\mathbb{Z}_p$, hence $\psi_b(e_a) \in \{0, 1\}$. We define the "support of a modulo p" to be $\mathrm{supp}(a) = \{b \in P : \psi_b(e_a) = 1\}$. Thus, $|\mathrm{supp}(a)| = 1$ iff $\psi(e_a)$ is a primitive idempotent of $\prod_{b \in P} \mathbb{Z}_p$. Our first objective is to establish the identity: $\sum_{a \in P^*} |\mathrm{supp}(a)| = |P|$.

5.1. LEMMA. (a) *If $a \neq b$, then* $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \varnothing$.

(b) *For any* $b \in P$, *there exists* $a \in P^*$ *(unique by part (a)) such that* $b \in \operatorname{supp}(a)$.

(c) *For any* $a \in P^*$, $\operatorname{supp}(a) \cap P^* = \{a\}$.

Proof. (a) Suppose $d \in \operatorname{supp}(a) \cap \operatorname{supp}(b)$, so that $\psi_d(e_a) = \psi_d(e_b) = 1$. Then $1 = \psi_d(e_a)\, \psi_d(e_b) = \psi_d(e_a e_b) = \psi_d(0) = 0$, a contradiction.

(b) If $b \notin \operatorname{supp}(a)$ for all $a \in P^*$, then $1 = \psi_b(1) = \psi_b(\sum_a e_a) = \sum_a \psi_b(e_a) = 0$, contradiction.

(c) Since $\psi_a(e_a) = \psi_a(\sum_{b \leqslant a} \lambda_{ab}[S_b]) = \sum_{b \leqslant a} \lambda_{ab} V_{ba} = \lambda_{aa} V_a = 1$, $a \in \operatorname{supp}(a)$. If also $b \in \operatorname{supp}(a) \cap P^*$, then $b \in \operatorname{supp}(a) \cap \operatorname{supp}(b) = \varnothing$. ∎

5.2. COROLLARY. $\sum_{a \in P^*} |\operatorname{supp}(a)| = |P|$.

Proof. By 5.1, $P = \dot{\bigcup}_{a \in P^*} \operatorname{supp}(a)$. ∎

5.3. LEMMA. *If* $a \in P^*$ *then the set* $\{[S_b]\, e_a : b \in \operatorname{supp}(a)\}$ *is a linearly independent subset of* $B_p e_a$.

Proof. For brevity denote $S = \operatorname{supp}(a)$. If $b \in S$, then $\psi_b(e_a) = 1$ implies $\sum_c \lambda_{ac} V_{cb} = 1$. Thus $[S_b]\, e_a = \sum_{c,d} \lambda_{ac} V_{cbd}[S_d] \equiv [S_b]$ (mod $\oplus_{c<b} \mathbb{Z}_p[S_c]$). Suppose there is a dependence relation $\sum_{b \in S} r_b[S_b]\, e_a = 0$, for some $r_b \in \mathbb{Z}_p$, and fix $g \in S$ maximal with $r_g \neq 0$. Thus $r_b \neq 0$ implies $g \nleqslant b$ if $b \neq g$, so that $[S_g] \notin \oplus_{c \leqslant b} \mathbb{Z}_p[S_c]$. Let $S' = \{b \in S \colon b \neq g,\ r_b \neq 0\}$, and let I be the ideal: $I = (\sum_{b \in S'} \sum_{c \leqslant b} \mathbb{Z}_p[S_c]) + (\sum_{d<g} \mathbb{Z}_p[S_d])$. By the computation above, $0 = \sum_{b \in S} r_b[S_b]\, e_a \equiv r_g[S_g] \not\equiv 0$ (mod I), a contradiction. ∎

5.4. THEOREM. *Let* $a \in P^*$. *Then* $\dim(B_p e_a) = |\operatorname{supp}(a)|$. *Moreover,* $\{[S_b]\, e_a : b \in \operatorname{supp}(a)\}$ *is a* $\mathbb{Z}_p$*-basis of* $B_p e_a$, *and* $\{[S_b]\, e_a \colon b \in \operatorname{supp}(a), b \neq a\}$ *is a* $\mathbb{Z}_p$*-basis of* $J(B_p e_a)$. *In particular,* $J_p = \oplus_{a \in P^*} \oplus_{b \in \operatorname{supp}(a), b \neq a} \mathbb{Z}_p[S_b]\, e_a$.

Proof. By 5.2 and 5.3 we have $\dim(B_p) = \sum_{a \in P^*} \dim(B_p e_a) \geqslant \sum_{a \in P^*} |\operatorname{supp}(a)| = |P| = \dim(B_p)$, and equality must hold thoughout. Thus, $\dim(B_p e_a) = |\operatorname{supp}(a)|$, and the second statement follows from 5.3. Next, note that if $b \in \operatorname{supp}(a)$ and $b \neq a$, then by 5.1(c), $b \notin P^*$. Thus by 3.2, $[S_b] \in \ker \psi = J_p$, so that $[S_b] e_a \in J(B_p e_a)$. Since $B_p e_a$ local, a dimension count now implies the third result. ∎

Because of 5.4, it would be helpful to have a formula for the "modular" idempotents of B_p. The techniques used by Gluck [4] fail to work in our situation; however, there is another approach which gives a formula for these idemptotents modulo the radical J_p. It is convenient to extend the ordering $\leqslant$ on P to a total ordering, so that $H_a \lesssim H_b$ implies $a \leqslant b$, but not necessarily conversely. We introduce some auxiliary matrices. Define $M=(V_{ab})_{a,b\in P}$, $M_p=(V_{ab})_{a,b\in P^*}$, and $M_p^*=(V_{ab})_{a\in P^*,b\in P}$. We consider M_p, M_p^* to have entries in $\mathbb{Z}_p$, so that their arithmetic is modulo p. In particular, M_p is nonsingular since it is lower triangular with nonzero diagonal entries. Define N_p to be the matrix product: $N_p=M_p^{-1}M_p^*=(W_{ab})_{a\in P^*,b\in P}$.

5.5. PROPOSITION. (a) $M_p^{-1}=(\lambda_{ab})_{a,b\in P^*}$.

(b) *For any* $a\in P^*$, $\psi(e_a)=(..., W_{ab},...)$. *Thus the* ath *row of* N_p *corresponds to the image under* ψ *of* e_a.

(c) $\mathrm{Supp}(a)=\{b\colon W_{ab}=1\}$. *In particular*, $\dim(B_pe_a)=\sum_{b\in P}W_{ab}$.

Proof. Let $c\in P^*$. Then by 5.1(c), $\delta_{ac}=\psi_c(e_a)=\sum_{b\in P}\lambda_{ab}V_{bc}$. However, if $b\notin P^*$, then $V_b=0$, whence $V_{bc}=0$, by 3.2. Thus the sum is over P^*, so that $\sum_{b\in P^*}\lambda_{ab}V_{bc}=\delta_{ac}$. Statement (a) follows.

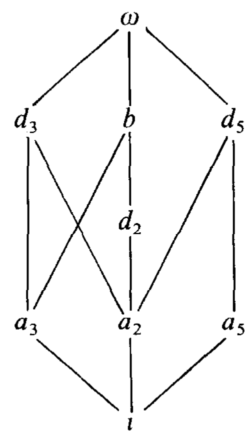
(b) It suffices to show that $\psi_b(e_a)=W_{ab}$. By part (a) we have: $\psi_b(e_a)=\psi_b(\sum_{c\in P}\lambda_{ac}[S_c])=\sum_{c\in P}\lambda_{ac}V_{cb}=\sum_{c\in P^*}\lambda_{ac}V_{cb}=W_{ab}$, since $V_{cb}=0$ if $c\notin P^*$.

(c) Immediate from part (b). ∎

Thus, modulo J_p, we can find the idempotents of B_p by inverting the matrix M_p. The following example illustrates the techniques involved.

6. EXAMPLE: $G=A_5$

The purpose of this section is to compute $B_2(A_5)$. The conjugacy classes of subgroups of A_5 are $\iota=(\{1\})$, $a_p=(\mathbb{Z}_p)$ $p=2, 3, 5$, $d_p=(D_p)$ $p=2, 3, 5$, $b=(A_4)$, and $\omega=(A_5)$. The lattice of $P(A_5)$ is given by the diagram:



We extend the ordering $\leqslant$ to a total ordering by setting $\iota < a_2 < a_3 < a_5 < d_2 < d_3 < d_5 < b < \omega$. It is a simple but tedious task to compute the double coset decompositions for pairs of subgroups of A_5, and then to compute the constants V_{ac} for all $a, c \in P(A_5)$. This information is given by the matrix

$$M = \begin{bmatrix} 60 & & & & & & & & \\ 30 & 2 & & & & & & \bigcirc & \\ 20 & 0 & 2 & & & & & & \\ 12 & 0 & 0 & 2 & & & & & \\ 15 & 3 & 0 & 0 & 3 & & & & \\ 10 & 2 & 1 & 0 & 0 & 1 & & & \\ 6 & 2 & 0 & 1 & 0 & 0 & 1 & & \\ 5 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We now consider the prime $p = 2$. Then $P^* = \{d_2, d_3, d_5, b, \omega\}$,

$$M_2 = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & \bigcirc & \\ 0 & 0 & 1 & & \\ 1 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad M_2^{-1} = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & \bigcirc & \\ 0 & 0 & 1 & & \\ 1 & 0 & 0 & 1 & \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore,

$$N_2 = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & \bigcirc & \\ 0 & 0 & 1 & & \\ 1 & 0 & 0 & 1 & \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & 0 & 1 & & \bigcirc & \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & & \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & 0 & 1 & & \bigcirc & \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

From 5.5(c) we have $\text{supp}(d_2) = \{\iota, a_2, d_2\}$, $\text{supp}(d_3) = \{a_3, d_3\}$, $\text{supp}(d_5) = \{a_5, d_5\}$, $\text{supp}(b) = \{b\}$, and $\text{supp}\ (\omega) = \{\omega\}$. Also, from 5.5(a), the primitive idempotents of $B_2(A_5)$ $(\text{mod}\, J_2(A_5))$ are $e_{d_p} \equiv [S_{d_p}]\ p = 2, 3, 5$, $e_b \equiv [S_b] + [s_{d_2}]$, and $e_\omega \equiv [S_\omega] + [S_b] + [S_{d_5}] + [S_{d_3}]$.

From 5.5(c), the dimensions of the local direct factors are 3, 2, 2, 1, 1, respectively. Moreover, $B_2(A_5) \cdot e_{d_2}$ has 2-nilpotent radical. This follows from direct computation, using the fact [6, Lemma 3] that

$J(B_2(A_5)\cdot e_{d_2})=\mathbb{Z}_p[S_\iota]\oplus\mathbb{Z}_p[S_{a_2}]$ (since D_2 is 2-Sylow in A_5). Algebras over finite fields of dimension 2 or 3 have been completely classified by Raghavendran [8, Theorems 10, 11]. In particular, a local ring of dimension 2 over $\mathbb{Z}_p$ is isomorphic with the ring of matrices $R_p(2)=\{(\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}): a, b\in\mathbb{Z}_p\}$, while a local ring of dimension 3 with 2-nilpotent radical is isomorphic with the ring of matrices

$$R_p(3)=\left\{\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a\end{pmatrix} : a, b, c\in\mathbb{Z}_p\right\}.$$

From this discussion it follows that: $B_2(A_5)\simeq\mathbb{Z}_2 \dotplus \mathbb{Z}_2 \dotplus R_2(2) \dotplus R_2(2) \dotplus R_2(3)$. In a similar fashion, the rings $B_3(A_5)$ and $B_5(A_5)$ can be computed. They are

$$B_3(A_5)\simeq\mathbb{Z}_3 \dotplus \mathbb{Z}_3 \dotplus \mathbb{Z}_3 \dotplus \mathbb{Z}_3 \dotplus \mathbb{Z}_3 \dotplus R_3(2) \dotplus R_3(2), \qquad \text{and}$$

$$B_5(A_5)\simeq\mathbb{Z}_5 \dotplus \mathbb{Z}_5 \dotplus \mathbb{Z}_5 \dotplus \mathbb{Z}_5 \dotplus \mathbb{Z}_5 \dotplus \mathbb{Z}_5 \dotplus \mathbb{Z}_5 \dotplus R_5(2).$$

7. The structure of the Support

This section is devoted to the problem of giving a non-combinatorial description of supp(a), for any $a\in P^*$. At the heart of this work, and what follows in Section 8, is the following lemma.

7.1. Lemma. *Suppose c, $d\in P$ are such that H_d is G-conjugate to a normal subgroup of H_c (notation: $H_d \trianglelefteq H_c$) of p-power index. Then for all $a\in P$, $V_{ad}\equiv V_{ac}$ (mod p).*

Proof. We may assume $H_d\subseteq H_c$. Take H_c-H_a and H_d-H_a double coset decompositions:

$$G=\dot{\bigcup_{i=1}^{V_{ac}}} H_c\tau_i H_a \cup \dot{\bigcup_{i=1}^{k}} H_c\alpha_i H_a$$

and

$$G=\dot{\bigcup_{i=1}^{V_{ad}}} H_d\sigma_i H_a \cup \dot{\bigcup_{i=1}^{m}} H_d\beta_i H_a,$$

where for all i, $H_c^{\tau_i}\subseteq H_a$, $H_d^{\sigma_i}\subseteq H_a$, $H_c^{\alpha_i}\nsubseteq H_a$, and $H_d^{\beta_i}\nsubseteq H_a$. Denote $S_i=H_d\sigma_i H_a$ and let $S=\{S_i\colon 1\leqslant i\leqslant V_{ad}\}$. Since H_d is normal in H_c, there is a well-defined H_c action on S given by $\sigma S_i=S_j$ if and only if $\sigma\sigma_i\in S_j$, all

$\sigma \in H_c$. Evidently, $H_d \subseteq \mathrm{Stab}_{H_c}(S_i)$ and $\mathrm{Stab}_{H_c}(S_i) = H_c \cap {}^{\sigma_i}H_a$. Thus by hypothesis, $|\mathrm{Orb}_{H_c}(S_i)| = (H_c : \mathrm{Stab}_{H_c}(S_i))$ is p-power for all i, with $|\mathrm{Orb}_{H_c}(S_i)| = 1$ if and only if $H_c^{\sigma_i} \subseteq H_a$.

Now observe that $V_{ac} = |\{\sigma_i : H_c^{\sigma_i} \subseteq H_a\}|$. To see this, define a map from $\{\sigma_i : H_c^{\sigma_i} \subseteq H_a\}$ to $\{\tau_j : 1 \leqslant j \leqslant V_{ac}\}$ by $\sigma_i \to \tau_j$ if and only if $\sigma_i \in H_c \tau_j H_a$. Note that $H_c \tau_i H_a = \tau_i H_a$ for $1 \leqslant i \leqslant V_{ac}$, and if $H_c^{\sigma_i} \leqslant H_a$, then $H_d \sigma_i H_a = \sigma_i H_a = \tau_j H_a$ for some j, $1 \leqslant j \leqslant V_{ac}$. Thus $\sigma_i \to \tau_j$ defines a bijection. Since S is a disjoint union of H_c orbits, $V_{ad} = |S| \equiv \sum_{H_c^{\sigma_i} \subseteq H_a} 1 = |\{\sigma_i : H_c^{\sigma_i} \subseteq H_a\}| = V_{ac} \pmod p$. ∎

7.2. Corollary. (Sylow) *If $p \mid |G|$, then the number of p-Sylow subgroups of G is congruent to one modulo p.*

Proof. Let c denote the class of the p-Sylow subgroups. Take $d = \iota$ and $a = c$ in 7.1. ∎

For any subgroup $H \leqslant G$ we let $O^p(H)$ denote the (unique) smallest normal subgroup of H of p-power index.

7.3. Theorem. *Let $a \in P^*$ and $d \in P$. The following are equivalent.*

(a) $d \in \mathrm{supp}(a)$.

(b) *There is a normal chain $H_d = H_0 \lhd H_1 \lhd \cdots \lhd H_n \sim H_a$ with each quotient a p-group.*

(c) $O^p(H_a) \lesssim H_d \lesssim H_a$.

Proof. (b) ⇒ (a) Induce on n. The case $n = 0$ being clear, assume that $n \geqslant 1$. Choose $c \in P$ with $H_1 \sim H_c$. By induction, $c \in \mathrm{supp}(a)$. Applying 7.1 we then have $\psi_d(e_a) = \sum_b \lambda_{ab} V_{bd} = \sum_b \lambda_{ab} V_{bc} = \psi_c(e_a) = 1$. Therefore, $d \in \mathrm{supp}(a)$.

(a) ⇒ (b) We define the chain inductively. Let $H_0 = H_d$, and assume we have constructed $H_0 \lhd H_1 \lhd \cdots \lhd H_k$ with each quotient a p-group. Choose $c \in P$ with $H_k \sim H_c$. Note that if $c \in P^*$ then by the implication (b) ⇒ (a) we have $d \in \mathrm{supp}(c) \cap \mathrm{supp}(a)$, thus $c = a$ and the chain is complete. If $c \notin P^*$ then $p \mid (N_G(H_c) : H_c)$, so let $H \leqslant G$ be such that H/H_c is p-Sylow in $N_G(H_c)/H_c$. Then $H_k \precsim H$, so there is a subgroup $H_{k+1} \sim H$ with $H_k \lhd H_{k+1}$ and the quotient a p-group.

(b) ⇔ (c) is standard, and we omit the proof. ∎

This Theorem has two interesting corollaries.

7.4. Corollary. *If p exactly divides $|G|$, then the local direct factors of B_p have $\mathbb{Z}_p$-dimension either one or two.*

Proof. By 4.3 and 5.4 it suffices to show that for any $a \in P^*$, $|\text{supp}(a)| = 1$ or 2. If p exactly divides $|H_a|$, then there can be at most one normal subgroup of H_a of index p, while there can be no such subgroups if p does not divide $|H_a|$. Apply 7.3. ∎

In fact, this Corollary is precise in the sense that if $p^2 \mid |G|$, then the local direct factor corresponding to a p-Sylow has dimension at least 3 (see [6]).

Using the characterization of dimension 2 algebras over $\mathbb{Z}_p$ as in Section 6 we find that if $|G|$ is square free, then for any prime p, $B_p(G) \simeq \mathbb{Z}_p^{(r)} \dotplus R_p(2)^{(s)}$, for some nonnegative integers r, s, depending on p, with $s > 0$ if and only if $p \mid |G|$. Since $R_p(2)$ has a unique proper ideal, it is trivial to check that $R_p(2)$ is self-injective. Since it is also Artinian, $R_p(2)$ is quasi-Frobenius. Moreover, the property of being quasi-Frobenius is invariant under finite direct products and scalar extensions. Hence we obtain in a more concrete fashion Corollary 2 of [6].

7.5. Corollary. *If G has square free order then for any field F, $F \otimes_{\mathbb{Z}} B(G)$ is a quasi-Frobenius F-algebra.*

8. Applications to Modular Representations

We now discuss $\mathbb{Z}_p$-representations of the group G, where we fix a prime p dividing $|G|$. For any G-set S, the permutation representation of S is integral, so by reducing modulo p we obtain a $\mathbb{Z}_p$-representation whose character will be dnoted by ξ_S. In particular if $S = S_a$ we denote by ξ_a the permutation character $1_{H_a}^G$ reduced modulo p. This correspondence yields a ring homomorphism $B(G) \to X(G, p)$, where $X(G, p)$ is the ring of $\mathbb{Z}_p$-characters. Since $pB(G)$ is contained in the kernel of this map, there is a ring homomorphism θ: $B_p(G) \to X(G, p)$, where θ satisfies $\theta([S]) = \xi_S$ for any G-set S. For each $\sigma \in G$, we let $a_\sigma \in P$ denote the class of the cyclic subgroup $\langle \sigma \rangle$ of G.

8.1. Proposition. (a) *Let $b \in P$ and $\sigma \in G$. Then $\xi_b(\sigma) = V_{ba_\sigma}$ (considered in $\mathbb{Z}_p$). In particular, if $b \notin P^*$, then $[S_b] \in \ker\theta$.*

(b) *Let $a \in P^*$ and denote $\mathcal{U}_a = \{\sigma \in G\colon O^p(H_a) \lesssim \langle \sigma \rangle \lesssim H_a\}$. Then $\theta(e_a)$ is the indicator function of the set $\mathcal{U}_a$. In particular, $e_a \notin \ker\theta$ if and only if H_a is p-hyperelementary.*

Proof. (a) Let $m = (1/|H_b|)\,|\{\tau \in G\colon \sigma\tau H_b = \tau H_b\}|$, then by definition $\xi_b(\sigma)$ equals m reduced modulo p. But $\sigma\tau H_b = \tau H_b$ if and only if $\langle \sigma \rangle^\tau \leqslant H_b$, so that $m = V_{ba_\sigma}$, which is the first statement. The second statement now follows from 3.2.

(b) Let $\sigma \in G$, then by 5.5 we have $\theta(e_a)(\sigma) = \theta(\sum_b \lambda_{ab}[S_b])(\sigma) = \sum_b \lambda_{ab} V_{ba_\sigma} = W_{aa_\sigma}$. Apply 5.5(c) and 7.3 to get the first statement. Finally, $e_a \notin \ker \theta$ if and only if $\mathscr{U}_a$ is nonempty if and only if $O^p(H_a)$ is cyclic if and only if H_a is p-hyperelementary. ▮

It will be helpful to have another description of the sets $\mathscr{U}_a$. Towards this we define, for any p-regular element x of G, the set $\mathscr{S}_x = \{\sigma \in G: \langle \sigma' \rangle \sim \langle x \rangle\}$, where σ' denotes the p-prime part of σ (see [5]).

8.2. Proposition. (a) *Let $a \in P^*$ be such that $e_a \notin \ker \theta$, and let $x \in G$ be any generator of the cyclic p'-group $O^p(H_a)$. Then $\mathscr{U}_a = \mathscr{S}_x$.*

(b) *Conversely, for any p-regular $x \in G$, there exists a unique $a \in P^*$ such that $\langle x \rangle \sim O^p(H_a)$, and for this a, $\mathscr{U}_a = \mathscr{S}_x$.*

Proof. (a) Choose $b \in P$ with $H_b \sim O^p(H_a)$. If $\sigma \in \mathscr{U}_a$, then $H_b \lesssim \langle \sigma \rangle$, hence $\langle \sigma' \rangle \sim H_b \sim \langle x \rangle$ implies $\sigma \in \mathscr{S}_x$. Conversely, if $\sigma \in \mathscr{S}_x$ then $H_b \sim \langle \sigma' \rangle \leqslant \langle \sigma \rangle$. Choose $c \in P^*$ with $a_\sigma \in \operatorname{supp}(c)$. Then by 7.3, $b \in \operatorname{supp}(c) \cap \operatorname{supp}(a)$, whence $c = a$ by 5.1. Thus $a_\sigma \leqslant a$, so $\langle \sigma \rangle \lesssim H_a$. By definition, $\sigma \in \mathscr{U}_a$.

(b) Say $\langle x \rangle \sim H_b$, some $b \in P$, and choose $a \in P^*$ with $b \in \operatorname{supp}(a)$. Obviously, $H_b \sim O^p(H_a)$, and by part (*a*), $\mathscr{U}_a = \mathscr{S}_x$. Uniqueness follows directly from 5.1. ▮

This proposition has a a consequence the following modular version of the main theorem of Gluck [5].

8.3. Theorem. *Let p be a prime dividing $|G|$. Then for any p-regular $x \in G$, the indicator function $I_{\mathscr{S}_x}$ is in the image of θ. In particular we may write $I_{\mathscr{S}_x} = \sum_H u_H 1_H^G$, $u_H \in \mathbb{Z}_p$, where the sum ranges over p-hyperelementary subgroups H satisfying $p \nmid (N_G(H): H)$ and $H \lesssim H_a$, where $a \in P^*$ is such that $\mathscr{U}_a = \mathscr{S}_x$.*

Proof. Let $a \in P^*$ be as in 8.2(b). Since $\mathscr{U}_a = \mathscr{S}_x$ is nonempty, H_a and all subconjugate subgroups are p-hyperelementary. Then, $I_{\mathscr{S}_x} = I_{\mathscr{U}_a} = \theta(e_a) = \theta(\sum_{b \leqslant a} \lambda_{ab}[S_b]) = \sum_{b \leqslant a, b \in P^*} \lambda_{ab} \xi_b$. A change of notation gives the result. ▮

We now describe the image of θ. Call a character $\chi: G \to \mathbb{Z}_p$ *p-constant* if whenever $\sigma, \tau \in G$ with $\langle \sigma' \rangle \sim \langle \tau' \rangle$, then $\chi(\sigma) = \chi(\tau)$. The essence of the Artin induction theorem is that the corresponding property holds for rational characters [1, Lemma 39.4], however, it is easy to discover $\mathbb{Z}_p$-characters that are not p-constant. Nevertheless, it is clear that sums and products of p-constant characters are again p-constant, and we obtain

8.4. THOREM. *The image of θ is the subring of $X(G, p)$ consisting of the p-constant characters.*

Proof. $\subseteq$ By 5.4 and 8.1(a), it suffices to show that $\theta([S_a]\, e_a)$ is p-constant for any $a \in P^*$. Choose $x \in G$ as in 8.2, and let $\sigma, \tau \in G$ be such that $\langle \sigma' \rangle \sim \langle \tau' \rangle$. If $\sigma \notin \mathscr{S}_x$ then plainly $\tau \notin \mathscr{S}_x$, hence $\theta([S_a]\, e_a)(\sigma) = \theta([S_a]\, e_a)(\tau) = 0$ in this case. Assume $\sigma, \tau \in \mathscr{S}_x$. Then by 7.1, $\theta([S_a]\, e_a)(\sigma) = \xi_a(\sigma) = V_{aa_\sigma} = V_{aa_{\sigma'}} = V_{aa_{\tau'}} = V_{aa_\tau} = \xi_a(\tau) = \theta([S_a]\, e_a)(\tau)$.

$\supseteq$ If χ is a p-constant character of G, then by definition χ is constant on each of the sets $\mathscr{S}_x$ for p-regular $x \in G$, hence on each of the sets $\mathscr{U}_a$, $a \in P^*$. Say $\chi(\mathscr{U}_a) = n_a \in \mathbb{Z}_p$ ($n_a := 0$ if $\mathscr{U}_a$ is empty). Then $\chi = \chi \cdot 1 = \chi \cdot \theta(\sum_{a \in P^*} e_a) = \sum_{a \in P^*} \chi \cdot I_{\mathscr{U}_a} = \sum_{a \in P^*} n_a I_{\mathscr{U}_a} = \theta(\sum_{a \in P^*} n_a e_a) \in$ image of θ. ∎

Finally, as a corollary of this result and 8.3, we obtain the following modular version of the Artin induction theorem.

8.5. THEOREM. *Any p-constant character $\chi: G \to \mathbb{Z}_p$ is a $\mathbb{Z}_p$-linear combination $\chi = \sum_H u_H 1_H^G$, where H ranges over those p-hyperelementary subgroups of G such that $p \nmid (N_G(H): H)$.*

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