GREEN FUNCTOR CONSTRUCTIONS IN THE THEORY OF ASSOCIATIVE ALGEBRAS.

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Jacobson, Eliot Thomas

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GREEN FUNCTOR CONSTRUCTIONS
IN THE THEORY OF
ASSOCIATIVE ALGEBRAS

by

Eliot Jacobson

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As members of the Final Examination Committee, we certify that we have read the dissertation prepared by Eliot Thomas Jacobson

entitled Green Functor Constructions in the Theory of Associative Algebras

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Robert C. Valentini — Date 4/7/83

Mohandas S. Cheema — Date 4/7/83

Larry C. Grove — Date 4/7/83

Richard S. Pierce — Date 4/7/83

Theodore Laetsch — Date 4/7/83

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copy of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Richard S. Pierce — 4/7/83
Dissertation Director — Date

STATEMENT BY AUTHOR

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SIGNED: Eliot T. Jacobson

Tiger got to hunt,

Bird got to fly;

Man got to sit and wonder, "why, why, why?"

Tiger got to sleep,

Bird got to land;

Man got to tell himself he understand.

-The Books of Bokonon

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ABSTRACT

Let G be a finite group. Given a contravariant, product preserving functor $F:G\text{-sets} \to AB$, we construct a Green-functor $A_F:G\text{-sets} \to CRNG$ which specializes to the Burnside ring functor when F is trivial. A_F permits a natural addition and multiplication between elements in the various groups $F(S)$, $S \in G$-sets. If G is the Galois group of a field extension L/K, and SEP denotes the category of K-algebras which are isomorphic with a finite product of subfields of L, then any covariant, product preserving functor $\rho:SEP \to AB$ induces a functor $F_\rho:G \to AB$, and thus the Green-functor A_ρ may be obtained. We use this observation for the case $\rho = Br$, the Brauer group functor, and show that $A_{Br}(G/G)$ is free on K-algebra isomorphism classes of division algebras with center in SEP. We then interpret the induction theory of Mackey-functors in this context. For a certain class of functors F, the structure of A_F is especially tractable; for these functors we deduce that $\mathbb{Q} \otimes_{\mathbb{Z}} A_F(G/G) \cong \Pi \mathbb{Q}F(S)$, where the product is over isomorphism class representatives of transitive G-sets. This allows for the computation of the prime ideals of $A_F(G/G)$, and for an explicit structure theorem for A_{Br}, when G is

the Galois group of a p-adic field. We finish by considering the case when $G = \mathrm{Gal}(L/\mathbb{Q})$, for an arbitrary number field L.

CHAPTER 1

INTRODUCTION

Let L/K be a finite Galois field extension, with Galois group G. Let $\mathcal{C}(L,K)$ be the category of K-algebras which are isomorphic with a finite product of subfields of L. We may then view the Brauer group as a covariant, additive functor $Br:\mathcal{C}(L,K) \to AB$, where AB denotes the category of abelian groups. Moreover, tensor product over K induces a multiplication among elements of the various groups $Br(A)$, $A \in \mathcal{C}(L,K)$. Since $\mathcal{C}(L,K)$ is anti-equivalent with the category $\hat{G}$ of finite G-sets, it is natural to ask if, given any contravariant functor $F:\hat{G} \to AB$ which transforms sums into products, there is a tensor product-like multiplication among elements of the groups $F(S)$, $S \in \hat{G}$. We outline such a construction (the details will be carried out in Chapter 3).

With G and F as above, for a G-set S define the category (G,S,F) to have as objects all triples (T,α,x), where $T \in \hat{G}$, $\alpha:T \to S$ is a G-map, and $x \in F(T)$. A morphism from (T,α,x) to (V,β,y) is a G-map $\phi:T \to V$ such that $\alpha = \beta\phi$, and $F(\phi)(y) = x$. Then (G,S,F) has direct sums and pullbacks, so we define $A_F(S)$ to be the associated Grothendieck ring $K_0(G,S,F)$. Multiplication in $A_F(S)$ essentially corresponds to the desired tensor product.

For example, if $F(T) = \{1\}$ for every G-set T, then $A_F(*)$ is the Burnside ring functor. In general, $A_F(*)$ is a Green-functor, and is, in particular, the left-adjoint to the natural forgetful functor $M \to M_*$ (see Chapter 4). If we apply this construction to the composite functor $\hat{G} \to C(L,K) \xrightarrow{Br} AB$, we obtain the Green-functor $A_{Br}(*)$. Especially, $A_{Br}(G)$ is free, with a basis corresponding to K-algebra ismorphism classes of division algebras with center in $C(L,K)$, where addition and multiplication are induced from direct product and tensor product (over K) respectively. The structure of $A_{Br}(G)$ can often be recovered from the following more general result.

For any G-set S, and $\alpha \in \mathrm{Aut}_G(S)$, the group automorphism $F(\alpha)$ induces a ring automorphism of the group algebra $\mathbb{Q}F(S)$. Let W_S denote the set of ring automorphisms of $\mathbb{Q}F(S)$ obtained in this way. Let $\mathbb{Q}F(S)^{W_S}$ denote the fixed ring. Our main structure theorem asserts that

$$\mathbb{Q} \otimes_{\mathbb{Z}} A_F(G) \cong \Pi \mathbb{Q}F(S)^{W_S}$$

the product being over isomorphism classes of transitive G-sets (see Chapter 5). Moreover, this isomorphism embeds

$A_F(G)$ into $\Pi\mathbb{Z}F(S)$, which then allows us to describe the prime ideals of $A_F(G)$ (see Chapter 6).

Chapter 7 is concerned with an alternate description of the ring $A_{Br}(G)$, which is much more manageable for applications. In particular, by applying the Mackey induction lemma we obtain the following cancellation theorem. If A and B are separable L-algebras such that $A \otimes_K L \cong B \otimes_K L$ as L-algebras, then $A \cong B$ as K-algebras.

We conclude by computing $\mathbb{Q} \otimes A_{Br}(E, \mathbb{Q}_p)$, when E is a Galois extension of the p-adic field $\mathbb{Q}_p$. This allows us to consider the ring $A_{Br}(N, \mathbb{Q})$, when N is a Galois extension of $\mathbb{Q}$. However, its computation leads us to the thorny problems of the isomorphism of adele rings, and the arithmetic equivalence of two number fields. These active areas of current research go beyond the intentions of this dissertation. Hence we must be content with an incomplete structure theorem for $A_{Br}(N, \mathbb{Q})$.

Finally, we must warn the reader that the proofs of many early results are quite computational. Most of the details are not omitted. Repeatedly the author has suppressed the temptation to skip over straight-forward proofs, often leaving a tedium of technicalities in the wake. The feeling is that this gives the reader a fair choice in the selection of proofs he wishes to work through, and the knowledge that someone, at least, has skinned his knuckles in checking all of the details.

CHAPTER 2

PRELIMINARY REMARKS

Throughout this chapter G will denote a fixed finite group. A G-set is a finite set on which G acts from the left. The category of all finite G-sets will be denoted by $\hat{G}$; its morphisms are set maps which commute with the action of G. Our objectives here are to define certain rings and functors associated with the category $\hat{G}$, and to set up some notation which will be useful to us throughout this dissertation.

The Burnside Ring

The set of isomorphism classes of finite G-sets becomes a commutative semi-ring with addition induced by disjoint union and multiplication by cartesian products. The Grothendieck ring constructed from this semi-ring is called the Burnside ring of G; it will be denoted $A(G)$. Thus, elements of $A(G)$ are formal differences $[S] - [T]$ where $S, T \in \hat{G}$. Moreover, $[S] + [T] = [S \dot{\cup} T]$ and $[S][T] = [S \times T]$.

Let $P = P(G)$ denote the set of all conjugacy classes of subgroups of G. For each $b \in P$, pick a representative H_b of b, and let S_b denote the transitive

G-set of cosets modulo H_b. For a, b, $c \in P$, let $V_{a,b,c}$ be the number of orbits in $S_a \times S_b$, under the diagonal action of G, which are isomorphic with S_c as G-sets. The following proposition collects some well known properties of $A(G)$.

Proposition 2.1. (a) Additively, $A(G)$ is free on the set $\{[S_a]:a \in P\}$, that is, $\{S_a:a \in P\}$ is a complete set of representatives of isomorphism classes of transitive G-sets.

(b) If $S, T \in \hat{G}$. then $[S] = [T]$ if and only if $S \cong T$ as G-sets.

(c) For a, $b \in P$, $[S_a][S_b] = \sum_{c\in P} V_{a,b,c}[S_c]$. Thus the $V_{a,b,c}$ are structure constants for $A(G)$.

The set P has a natural partial ordering, where we set $a \leq b$ precisely when H_a is subconjugate to H_b (denoted $H_a \lesssim H_b$). As in Solomon (1967), $\mathbb{Q}A(G) = \mathbb{Q} \otimes_{\mathbb{Z}} A(G)$ has primitive idempotents $\{e_a:a \in P\}$, where $e_a = \sum_{b\leq a} \lambda_{b,a}[S_b]$ for suitable constants $\lambda_{b,a} \in \mathbb{Q}$. We shall define $\lambda_{b,a} = 0$ if $b \not\leq a$ so that we may write $e_a = \sum_{b\in P} \lambda_{b,a}[S_b]$. It follows that $\sum_{c\in P} e_c = 1_{A(G)}$, and $e_a e_b = \delta_{ab} e_a$, for all a, $b \in P$. We summarize some known results on the constants $\lambda_{a,b}$ and $V_{a,b,c}$.

Proposition 2.2. (a) for any $a \in P$, $V_{a,a,a} = \lambda^{-1}_{a,a} = [N_G(H_a):H_a]$.

(b) For any $a \in P$, $|G|e_a \in A(G)$. Thus $|G| \cdot \lambda_{b,a} \in \mathbf{Z}$ for all a, $b \in P$.

(c) For any a, b, $c \in P$, $V_{a,b,c} = 0$ unless both $c \leq a$ and $c \leq b$.

We just remark that 2.2(b) can be strengthened to the statement $|N_G(H_a)| \cdot e_a \in A(G)$, for any $a \in P$, by the idempotent formula of Gluck (1981). However, we will have no use for this extra information. For brevity we shall denote $V_a = V_{a,a,a}$ and $V_{a,b} = V_{a,b,b}$, all a, $b \in P$. Fundamental to our later work are the following propositions relating the constants $V_{a,b,c}$ and $\lambda_{a,b}$.

Proposition 2.3. Let $a < b \in P$. Then for all $d \in P$, $\sum_{c \in P} \lambda_{c,b} V_{a,c,d} = 0$.

Proof. Note that
$$\begin{aligned} 0 &= e_a \cdot e_b \\ &= \sum_{c,d} \lambda_{c,a}\lambda_{d,b}[S_c][S_d] \\ &= \sum_{c,d,e} \lambda_{c,a}\lambda_{d,b}V_{c,d,e}[S_e] \\ &= \sum_e \Big(\sum_{c,d} \lambda_{c,a}\lambda_{d,b}V_{c,d,e}\Big)[S_e]. \end{aligned}$$

By 2.1(a) it follows that

$$(*) \qquad \sum_{c,d} \lambda_{c,a}\lambda_{d,b}V_{c,d,e} = 0 \quad \text{for all } e \in P.$$

We establish the required formula by induction on $a \in P$ with respect to the partial order $\leq$. If $a = \{1\}$ (the unique minimal element) then since $\lambda_{c,a} = 0$ if $c \not\leq a$, (*) becomes $\lambda_{a,a}\sum_{d}\lambda_{d,b}V_{a,d,e} = 0$, all $e \in P$. Since $\lambda_{a,a} \neq 0$ by 2.2(a), this starts the induction. Assume that $a \neq \{1\}$, and that whenever $c < a$, and $e \in P$, then $\sum_{d}\lambda_{d,b}V_{c,d,e} = 0$. By (*), for any $e \in P$ we have

$$0 = \lambda_{a,a}\sum_{d}\lambda_{d,b}V_{a,d,e} + \sum_{c<a}\lambda_{c,a}(\sum_{d}\lambda_{d,b}V_{c,d,e})$$

$$= \lambda_{a,a}\sum_{d}\lambda_{d,b}V_{a,d,e} \qquad \text{(by induction)}.$$

Since $\lambda_{a,a} \neq 0$, $\sum_{d}\lambda_{d,b}V_{a,d,e} = 0$ for all $e \in P$, as claimed. □

Proposition 2.4. Let $a, b \in P$ with $b \not\leq a$. Then $[S_a]e_b = 0$.

Proof. Note that $[S_a]e_b = [S_a]e_b e_b$

$$= \sum_{c \leq b}\lambda_{c,b}[S_a][S_c]e_b$$

$$= \sum_{c \leq b}\sum_{d \leq a,c}\lambda_{c,b}V_{a,c,d}[S_d]e_b.$$

Thus it suffices to show $[S_d]e_b = 0$ whenever $c \leq b$ and $d \leq a$, c. The condition $b \not\leq a$ then forces $d < b$, so we may as well assume $a < b$ to begin with. Then, by the above computation and Proposition 2.3,

$$[S_a]e_b = \sum_d (\sum_c \lambda_{c,b} V_{a,c,d})[S_d]e_b = 0. \qquad \square$$

<u>Proposition 2.5</u>. (a) For any $a \in P$, $e_a = V_a^{-1}[S_a]e_a$.

(b) If a, $c \in P$, then $\sum_{b \in P} \lambda_{b,a} V_{a,b,c} = \lambda_{c,a} V_a$.

<u>Proof</u>. (a)

$$\begin{aligned} e_a &= e_a \cdot e_a \\ &= \sum_{b \leq a} \lambda_{b,a}[S_b]e_a \\ &= \lambda_{a,a}[S_a]e_a, \quad \text{by 2.4} \\ &= V_a^{-1}[S_a]e_a, \quad \text{by 2.2(a).} \end{aligned}$$

(b) By (a),

$$\begin{aligned} e_a &= V_a^{-1}[S_a]e_a \\ &= V_a^{-1}\sum_b \lambda_{b,a}[S_a][S_b] \\ &= V_a^{-1} \sum_{b,c} \lambda_{b,a} V_{a,b,c}[S_c] \\ &= \sum_c (V_a^{-1}\sum_b \lambda_{b,a} V_{a,b,c})[S_c]. \end{aligned}$$

Comparing coefficients and applying 2.1(a) yields

$$\lambda_{c,a} = V_a^{-1}\sum_b \lambda_{b,a} V_{a,b,c}$$

as claimed. $\square$

We must indicate some notational conventions in the category $\hat{G}$. We shall always use a subscripted K to denote an inclusion map in $\hat{G}$. For example, if S, $T \in \hat{G}$ we may denote by K_S the canonical inclusion of S into $S \dot{\cup} T$. If $a \in P$, we may use the notation $K_a : S_a \to S_a \dot{\cup} T$. Similarly, we shall always use a subscripted π to denote a projection map in $\hat{G}$. Thus one might see $\pi_S : S \times T \to S$, or $\pi_a : S_a \times T \to S_a$. The point is that the subscript will always be sufficient for the reader to deduce the map, explicit mention of domains and ranges will seldom be given.

Mackey-Functors and Frobenius-Functors

Various equivalent definitions of Mackey-functors, Frobenius-functors and Green-functors have appeared over the last few years. Our definitions roughly coincide with those of Kuchler (1970).

Definition 2.6. A Mackey-functor on G is a bifunctor $M = (M^*, M_*) : \hat{G} \to AB$, where M^* is covariant, M_* is contravarient, M^* and M_* agree on objects, such that the following conditions are fulfilled by M.

(a) If

$$\begin{tikzcd} X \arrow[r, "\psi_2"] \arrow[d, "\psi_1"'] & X_2 \arrow[d, "\phi_2"] \\ X_1 \arrow[r, "\phi_1"'] & Y \end{tikzcd}$$

is a pullback diagram in $\hat{G}$, then the diagram

$$\begin{tikzcd} M(X) \arrow[r, "M^*(\psi_2)"] & M(X_2) \\ M(X_1) \arrow[u, "M_*(\psi_1)"] \arrow[r, "M^*(\phi_1)"'] & M(Y) \arrow[u, "(M_*(\phi_2)"'] \end{tikzcd}$$

commutes.

(b) If S_1, $S_2 \in \hat{G}$ with inclusions $K_i : S_i \to S_1 \dot{\cup} S_2$, then the homomorphisms $M_*(K_i) : M(S_1 \dot{\cup} S_2) \to M(S_i)$ induce an isomorphism $M_*(K_1) \times M_*(K_2) : M(S_1 \dot{\cup} S_2) \to M(S_1) \times M(S_2)$.

For a G-map $\alpha : S \to T$, we will denote $\alpha^* = M^*(\alpha)$ and $\alpha_* = M_*(\alpha)$ when no confusion will arise.

<u>Definition 2.7</u>. A <u>Frobenius-functor</u> on G is a bifunctor $M = (M^*, M_*) : \hat{G} \to AB$, with M^* covariant, M_* contravariant, M^* and M_* coincide on objects, such that M satisfies the following.

(a) For each G-set S, M(S) is a commutative ring with 1.

(b) For each G-map $\alpha : S \to T$. $\alpha_* : M(T) \to M(S)$ is a ring homomorphism (preserving unit).

(c) For each G-map $\alpha : S \to T$ we may view M(S) as a M(T)-module via α_*. We then require $\alpha^* : M(S) \to M(T)$ to be a $M(T)$-module homomorphism. Thus for any $s \in M(S)$, $t \in M(T)$, we have $\alpha^*(\alpha_*(t) \cdot s) = t \cdot \alpha^*(s)$ (Frobenius reciprocity).

<u>Definition 2.8</u>. A <u>Green-functor</u> on G is a bifunctor $M = (M^*, M_*): \hat{G} \to AB$ which is simultaneously both a Mackey-functor and a Frobenius-functor.

Finally, we wish to record two elementary properties of these functors which will be useful in Chapter 4.

<u>Proposition 2.9</u>. If $M: \hat{G} \to AB$ is a Mackey-functor, and if $\alpha: S \to T$ is an isomorphism of G-sets, then α^* and α_* are inverse isomorphisms.

<u>Proof</u>. Since α is an isomorphism, the diagrams

$$\begin{tikzcd} S \arrow[r, "1"] \arrow[d, "1"'] & S \arrow[d, "\alpha"] \\ S \arrow[r, "\alpha"'] & T \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & T \arrow[d, "1"] \\ T \arrow[r, "1"'] & T \end{tikzcd}$$

are pullbacks in $\hat{G}$. Applying 2.6(a) to each diagram yields $\alpha_*\alpha^* = 1_{M(S)}$ and $\alpha^*\alpha_* = 1_{M(T)}$.

<u>Proposition 2.10</u>. Let S_1, $S_2 \in \hat{G}$, with inclusion maps $K_i: S_i \to S_1 \dot{\cup} S_2$. If $M: \hat{G} \to AB$ is a Mackey-functor, then $K^*_1 K_{1*} + K^*_2 K_{2*}: M(S_1 \dot{\cup} S_2) \to M(S_1 \dot{\cup} S_2)$ is the identity map.

<u>Proof</u>. The diagrams

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & S_2 \arrow[d, "K_2"] \\ S_1 \arrow[r, "K_1"'] & S_1 \dot{\cup} S_2 \end{tikzcd} \qquad \begin{tikzcd} S_1 \arrow[r, "1"] \arrow[d, "1"'] & S_1 \arrow[d, "K_1"] \\ S_1 \arrow[r, "K_1"'] & S_1 \dot{\cup} S_2 \end{tikzcd}$$

are pullbacks. Then, by 2.6(a), $K_{2*}K_1^* = 0$, and $K_{1*}K_1^* = 1$. Similarly, $K_{1*}K_2^* = 0$, and $K_{2*}K_2^* = 1$. If $x \in M(S_1 \dot{\cup} S_2)$, then $K_{1*}(K_1^*K_{1*}(x) + K_2^*K_{2*}(x)) = K_{1*}(x)$, and $K_{2*}(K_1^*K_{1*}(x) + K_2^*K_{2*}(x)) = K_{2*}(x)$. By 2.6(b), $K_1^*K_{1*}(x) + K_2^*K_{2*}(x) = x$. □

CHAPTER 3

THE F-BURNSIDE RING

In this chapter we shall construct one of the main objects of our study. We then prove a few elementary results which will be essential for later applications.

The Basic Construction

Let G be a finite group, fixed throughout the remainder of this chapter. Let $F:\hat{G} \to AM$ be a contravariant functor, where AM denotes the category of abelian monoids. For a G-map $\alpha:S \to T$, we shall denote $\alpha^0 = F(\alpha):F(T) \to F(S)$. If given any two G-sets, S_1, S_2, with inclusions $K_i:S_i \to S_1 \dot{\cup} S_2$, the induced map $K_1^0 \times K_2^0:F(S_1 \dot{\cup} S_2) \to F(S_1) \times F(S_2)$ is an isomorphism, then we shall call F <u>additive</u>. For an additive functor F and elements $x \in F(S_1)$, $y \in F(S_2)$, we introduce the notation $x \dot{+} y$ to denote the unique element of $F(S_1 \dot{\cup} S_2)$ satisfying $K_1^0 \times K_2^0(x \dot{+} y) = (x,y)$. Thus $K_1^0(x \dot{+} y) = x$, and $K_2^0(x \dot{+} y) = y$. For the remainder of this section, assume we have a fixed additive contravariant functor $F:\hat{G} \to AM$.

For any G-set S, we form the category (G,S,F) as follows:

Objects: Triples (T,ϕ,x) where $T \in \hat{G}$, $\phi:T \to S$ is a G-map, and $x \in F(T)$.

Morphisms: A morphism $(T,\phi,x) \to (V,\psi,y)$ is a G-map $\alpha:T \to V$ such that $\phi = \psi\alpha$ and $\alpha^0(y) = x$.

Given (T,ϕ,x), (V,ψ,y) in (G,S,F), define $(T,\phi,x) \oplus (V,\psi,y)$ to equal $(T \dot{\cup} V, \phi \dot{\cup} \psi, x \dotplus y)$. The later is an object of (G,S,F) since F is additive. It is routine to check that $\oplus$ is a categorical coproduct for (G,S,F). Also, by considering the pullback diagram

$$\begin{tikzcd} Tx_S V \arrow[r, "\pi_V"] \arrow[d, "\pi_T"'] & V \arrow[d, "\psi"] \\ T \arrow[r, "\phi"'] & S \end{tikzcd}$$

in $\hat{G}$, we may define $(T,\phi,x)x_S(V,\psi,y)$ to equal $(Tx_S V, \phi x_S \psi, \pi_T^0(x) \cdot \pi_V^0(y))$.

The operations $\oplus$ and x_S satisfy all of the necessary identities (check!) to form the half ring $A_F^+(S)$ of isomorphism classes of objects in (G,S,F), with addition induced by $\oplus$ and multiplication by x_S. We denote the associated Grothendieck ring by $A_F(S)$, and refer to this ring as the F-<u>Burnside</u> <u>ring</u> of G-sets over S. We let $[T,\phi,x]$ denote the image of (T,ϕ,x) in $A_F(S)$. The following lemma collects some standard results about the

Grothendieck group of a category with product, as applied to $A_F(S)$ (Bass 1968, pp. 344-47).

<u>Lemma 3.1</u>. (a) Each element of $A_F(S)$ has the form $[T,\phi,x] - [V,\psi,y]$, for suitable (T,ϕ,x), (V,ψ,y) in (G,S,F).

(b) $[T,\phi,x] + [V,\psi,y] = [T \dot{\cup} V, \phi \dot{\cup} \psi, x \dotplus y]$, and $[T,\phi,x] \cdot [V,\psi,y] = [Tx_S V, \phi x_S \psi, \pi_T^0(x) \cdot \pi_V^0(y)]$.

(c) $[T,\phi,x] = [V,\psi,y]$ if and only if there exists (U,λ,z) in (G,S,F) such that $(T \dot{\cup} U, \phi \dot{\cup} \lambda, x \dotplus z) \cong (V \dot{\cup} U, \psi \dot{\cup} \lambda, y \dotplus z)$ in (G,S,F).

<u>A Cancellation Theorem in (G,S,F)</u>

The goal of this section is a strengthening of 3.1(c). Fixed throughout the present discussion is a G-set S, and an additive contravariant functor $F:\hat{G} \to AM$.

<u>Lemma 3.2</u>. Suppose (T,α,x), (V,β,y) and (W,γ,z) are in (G,S,F), and that T is a transitive G-set. If $(T,\alpha,x) \ominus (V,\beta,y) \cong (T,\alpha,x) \ominus (W,\gamma,z)$ in (G,S,F), then $(V,\beta,y) \cong (W,\gamma,z)$.

<u>Proof</u>. By hypothesis, $(T \dot{\cup} V, \alpha \dot{\cup} \beta, x \dotplus y) \cong (T \dot{\cup} W, \alpha \dot{\cup} \gamma, x \dotplus z)$, so there is a G-isomorphism $\phi:T \dot{\cup} V \to T \dot{\cup} W$ with $\alpha \dot{\cup} \beta = (\alpha \dot{\cup} \gamma) \circ \phi$ and $\phi^0(x \dotplus z) = x \dotplus y$. Since T is transitive, and $\phi(T)$ is non-empty, either $\phi(T) = T$ or $\phi(T) \subseteq W$. We consider these cases separately.

Case 1) $\phi(T) = T$. Then $\phi(V) = W$. Write $\phi = \mu \dot{\cup} \lambda$, with $\mu = \phi|_T : T \to T$, and $\lambda = \phi|_V : V \to W$. Let $K_V : V \to T \dot{\cup} V$ and $K_W : W \to T \dot{\cup} W$ be inclusions. Clearly, $\phi K_V = K_W \lambda$. Thus, $\lambda^0(z) = \lambda^0 K_W^0(x \dotplus z) = K_V^0 \phi^0(x \dotplus z) = K_V^0(x \dotplus y) = y$. Moreover, if $v \in V$, then $\gamma\lambda(v) = (\alpha \dot{\cup} \gamma)\phi(v) = (\alpha \dot{\cup} \beta)(v) = \beta(v)$, that is, $\gamma\lambda = \beta$. It follows that $\lambda : (V,\beta,y) \to (W,\gamma,z)$ is an isomorphism, finishing this case.

Case 2) $\phi(T) \subseteq W$, and therefore $T \subseteq \phi(V)$. Hence we may write $V = T_1 \dot{\cup} V'$, where $\phi(T_1) = T$, and $W = T_2 \dot{\cup} W'$, where $\phi(T) = T_2$. By additivity of F, write $y = x_1 \dotplus y'$ and $z = x_2 \dotplus z'$, where $x_i \in F(T_i)$, $y' \in F(V')$ and $z' \in F(W')$. We may also write $\phi = \mu \dot{\cup} \lambda \dot{\cup} \delta$, with $\mu = \phi|_T : T \to T_2$, $\lambda = \phi|_{T_1} : T_1 \to T$, and $\delta = \phi|_{V'} : V' \to W'$ all isomorphisms. As in Case 1, it follows that $\mu^0(x_2) = x$, $\lambda^0(x) = x_1$ and $\delta^0(z') = y'$. Define $\psi : V \to W$ to be $\mu\lambda \dot{\cup} \delta$. Then $\psi^0(z) = (\mu\lambda \dot{\cup} \delta)^0(x_2 \dotplus z') = (\mu\lambda)^0(x_2) \dotplus \delta^0(z') = \lambda^0\mu^0(x_2) \dotplus \delta^0(z') = x_1 \dotplus y' = y$. Finally, to show $\beta = \gamma\psi$, let $v \in V$. Let $K_T : T \to T \dot{\cup} V$ be inclusion, so that $\mu = \phi K_T$. If $v \in T_1$, then $\gamma\psi(v) = \gamma\mu\lambda(v) = \gamma\phi K_T\lambda(v) = (\alpha \dot{\cup} \gamma)\phi(K_T\lambda(v)) = (\alpha \dot{\cup} \beta)(K_T\lambda(v)) = \alpha\lambda(v) = (\alpha \dot{\cup} \gamma)\phi(v) = (\alpha \dot{\cup} \beta)(v) = \beta(v)$. If $v \in V'$ then $\gamma\psi(v) = \gamma\delta(v) = (\alpha \dot{\cup} \gamma)\phi(v) = (\alpha \dot{\cup} \beta)(v) = \beta(v)$. Thus $\psi : (V,\beta,y) \to (W,\gamma,z)$ is an isomorphism. $\square$

<u>Theorem 3.3</u>. Suppose (T,α,x), (V,β,y), $(W,\gamma,z) \in (G,S,F)$ satisfy $(T,\alpha,x) \oplus (V,\beta,y) \cong (T,\alpha,x) \oplus (W,\gamma,z)$. Then $(V,\beta,y) \cong (W,\gamma,z)$.

<u>Proof</u>. Write $T = \dot{\bigcup}_{i=1}^{n} T_i$, where each T_i is a transitive G-set, and let $\alpha_i = \alpha|_{T_i} : T_i \to S$. By additivity of F, there exists $x_i \in F(T_i)$ so that $(T,\alpha,x) \cong \bigoplus_{i=1}^{n} (T_i,\alpha_i,x_i)$. By the lemma, we may cancel the (T_i,α_i,x_i) one at a time yielding the result. □

<u>Corollary 3.4</u>. $[V,\beta,y] = [W,\gamma,z]$ in $A_F(S)$ if and only if $(V,\beta,y) \cong (W,\gamma,z)$ in (G,S,F).

<u>A_F is a Green-Functor</u>

We shall now establish the fundamental fact that A_F is a Green-functor. More precisely, fix an additive contravariant functor $F:\hat{G} \to AM$, then we shall define covariant and contravariant morphism maps which turn the correspondence $S \to A_F(S)$ into the object map of a Green-functor.

Suppose S, $T \in \hat{G}$, and $\alpha:S \to T$ is a G-map. Then the map $(V,\phi,x) \to [V,\alpha\phi,x]$ from (G,S,F) to $A_F(T)$ respects isomorphism in (G,S,F) and is additive (preserves $\oplus$). Thus there is an induced group homomorphism $\alpha^* = A_F^*(\alpha): A_F(S) \to A_F(T)$ satisfying $\alpha^*([V,\phi,x]) = [V,\alpha\phi,x]$, all $[V,\phi,x] \in A_F(S)$. To describe a map $\alpha_* = A_{F*}(\alpha):A_F(T) \to A_F(S)$, note that for any $(W,\psi,y) \in (G,T,F)$ we have a pullback diagram

$$\begin{tikzcd} W\times_T S \arrow[r, "\pi_S"] \arrow[d, "\pi_W"'] & S \arrow[d, "\alpha"] \\ W \arrow[r, "\psi"'] & T \end{tikzcd},$$

hence an element $[W\times_T S, \pi_S, \pi_W^0(y)]$ of $A_F(S)$.

<u>Proposition 3.5</u>. Given any G-map $\alpha: S \to T$, the correspondence $(W,\psi,y) \to [W\times_T S, \pi_S, \pi_W^0(y)]$ induces a ring homormophism $\alpha_*: A_F(T) \to A_F(S)$ satisfying $\alpha_*([W,\psi,y]) = [W\times_T S, \pi_S, \pi_W^0(y)]$, for all $[W,\psi,y] \in A_F(T)$.

<u>Proof</u>. Define $\lambda: (G,T,F) \to A_F(S)$ by $\lambda(W,\psi,y) = [W\times_T S, \pi_S, \pi_W^0(y)]$. It suffices to show that λ is constant on isomorphism classes, and that λ respects $\oplus$ and $\times_T$ (thus λ induces α_* above). Fix (V,ϕ,x), (W,ψ,y) in (G,T,F).

i) If $(V,\phi,x) \cong (W,\psi,y)$, Choose $\beta: V \to W$, a G-isomorphism, with $\phi = \psi\beta$ and $\beta^0(y) = x$. If $(v,s) \in V\times_T S$, then $\alpha(s) = \phi(v) = \psi(\beta(v))$, so that $(\beta(v),s) \in W\times_T S$. Thus the map $\gamma: V\times_T S \to W\times_T S$ given by $\gamma(v,s) = (\beta(v),s)$ is a G-isomorphism. Plainly, $\pi_S\gamma = \pi_S$ and $\pi_W\gamma = \beta\pi_V$. Thus $\gamma^0\pi_W^0(y) = \pi_V^0\beta^0(y) = \pi_V^0(x)$. It follows that $\gamma: (V\times_T S, \pi_S, \pi_V^0(x))$

$\to (Wx_TS,\pi_S,\pi^0_W(y))$ is an isomorphism, hence $[Vx_TS,\pi_S,\pi^0_V(x)] = [Wx_TS,\pi_S,\pi^0_W(y)]$ in $A_F(S)$.

ii) To see that λ respects $\dot{+}$, it suffices to show that $(Vx_TS \dot{U} Wx_TS,\pi_S \dot{U} \pi_S,\pi^0_V(x) \dot{+} \pi^0_W(y)) \cong ((V \dot{U} W)x_TS,\pi_S,\pi^0_{V\dot{U}W}(x \dot{+} y))$. Define $\gamma: Vx_TS \dot{U} Wx_TS \to (V \dot{U} W)x_TS$ to be the identity. Evidently, γ is an isomorphism such that $\pi_S\gamma = \pi_S \dot{U} \pi_S$. We need $\gamma^0\pi^0_{V\dot{U}W}(x \dot{+} y) = \pi^0_V(x) \dot{+} \pi^0_W(y)$. Let $K_V: Vx_TS \to Vx_TS \dot{U} Wx_TS$ and $j_V: V \to V \dot{U} W$ be inclusions. Plainly, $\pi_{V\dot{U}W}\gamma K_V = j_V\pi_V$. Thus $K^0_V(\gamma^0\pi^0_{V\dot{U}W}(x \dot{+} y)) = \pi^0_V j^0_V(x \dot{+} y) = \pi^0_V(x)$. Similarly, if $K_W: Wx_TS \to Vx_TS \dot{U} Wx_TS$ is inclusion, then $K^0_W(\gamma^0\pi^0_{V\dot{U}W}(x \dot{+} y)) = \pi^0_W(y)$. By the additive of F, $\gamma^0\pi^0_{V\dot{U}W}(x \dot{+} y) = \pi^0_V(x) \dot{+} \pi^0_W(y)$.

iii) To show that λ respects x_S, it suffices to show that $((Vx_TW)x_TS,\ \pi_S,\ \pi^0_{Vx_TW}(\pi^0_V(x)\ \cdot\ \pi^0_W(y))) \cong ((Vx_TS)x_S(Wx_TS),\ \pi_Sx_S\pi_S,\ \pi^0_{Vx_TS}(\pi^0_V(x))\ \cdot\ \pi^0_{Wx_TS}(\pi^0_W(y)))$. Define $\gamma: (Vx_TW)x_TS \to (Vx_TS)x_S(Wx_TS)$ by $\gamma((v,w),s) = ((v,s),(w,s))$. Then γ is the canonical isomorphism, and it follows easily that $\pi_S = (\pi_Sx_S\pi_S) \circ \gamma$. Moreover, the following diagram commutes

$$
\begin{tikzcd}
(V\times_T W)\times_T S \arrow[r, "\gamma"] \arrow[d, "\pi_{V\times_T W}"'] & (V\times_T S)\times_S(W\times_T S) \arrow[r, "\pi_{W\times_T S}"] \arrow[dd, "\pi_{V\times_T S}"] & W\times_T S \arrow[d, "\pi_W"] \\
V\times_T W \arrow[rr, "\pi_W"] \arrow[d, "\pi_V"'] & & W \\
V & V\times_T S \arrow[l, "\pi_V"] &
\end{tikzcd}
$$

Thus, $\gamma^0(\pi^0_{V\times_T S}(\pi^0_V(x)) \cdot \pi^0_{W\times_T S}(\pi^0_W(y))) = (\pi_V\pi_{V\times_T S}\gamma)^0(x)$
$\cdot\ (\pi_W\pi_{W\times_T S}\gamma)^0(y) = (\pi_V\pi_{V\times_T W})^0(x) \cdot (\pi_W\pi_{V\times_T W})^0(y)$
$= \pi_{V\times_T W}(\pi^0_V(x) \cdot \pi^0_W(y))$. □

Theorem 3.6. Let G be a finite group, and $F:\hat{G} \to AM$ be an additive contravariant functor. Then $A_F = (A_F^*, A_{F*})$ is a Green-functor.

Proof. We must verify the axioms 2.6(a), (b), and 2.7(a), (b), (c).

Axiom 2.6(a). Let

$$
\begin{tikzcd}
X \arrow[r, "\psi_2"] \arrow[d, "\psi_1"'] & X_2 \arrow[d, "\phi_2"] \\
X_1 \arrow[r, "\phi_1"'] & Y
\end{tikzcd}
$$

be a pullback diagram in $\hat{G}$. We must show that $\phi_{2*}\phi_1^* = \psi_2^*\psi_{1*}$: $A_F(X_1) \to A_F(X_2)$. Let $[S,\alpha,x] \in A_F(X_1)$. Then

$\phi_{2*}\phi_1^*([S,\alpha,x]) = [X_2 x_Y S, \pi_{X_2}, \pi_S^0(x)]$, and $\psi_2^*\psi_{1*}([S,\alpha,x])$ $= [X x_{X_1} S, \psi_2\pi_X, \tilde{\pi}_S^0(x)]$, where the pullback diagrams

$$\begin{tikzcd} X_2 x_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_{X_2}"'] & S \arrow[d, "\phi_1\alpha"] \\ X_2 \arrow[r, "\phi_2"'] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X x_{X_1} S \arrow[r, "\tilde{\pi}_S"] \arrow[d, "\pi_X"'] & S \arrow[d, "\alpha"] \\ X \arrow[r, "\psi_1"'] & X_1 \end{tikzcd}$$

explain our notation. Define $\gamma: X x_{X_1} S \to X_2 x_Y S$ by $\gamma(x,s) = (\psi_2(x),s)$. Using the fact that $X \cong X_1 x_Y X_2$, it is easy to see that γ is an isomorphism of G-sets.

It is equally evident that $\psi_2\pi_X = \pi_{X_2}\gamma$. Since $\tilde{\pi}_S = \pi_S\gamma$, it follows that $\gamma^0\pi_S^0(x) = \tilde{\pi}_S^0(x)$. Thus $\gamma: (X x_{X_1} S, \psi_2\pi_X, \tilde{\pi}_S^0(x)) \to (X_2 x_Y S, \pi_{X_2}, \pi_S^0(x))$ is an isomorphism. Thus, $\phi_{2*}\phi_1^* = \psi_2^*\psi_{1*}$.

Axiom 2.6(b). Let S_1, S_2 be G-sets, and let K_i: $S_i \to S_1 \dot{U} S_2$ be the inclusion maps. We show that $K_{1*} \times K_{2*}$: $A_F(S_1 \dot{U} S_2) \to A_F(S_1) \times A_F(S_2)$ is an isomorphism by exhibiting its inverse. We define $\beta: A_F(S_1) \times A_F(S_2) \to A_F(S_1 \dot{U} S_2)$ by $\beta([T_1,\phi_1,X_1],[T_2,\phi_2,X_2]) = [T_1 \dot{U} T_2, \phi_1 \dot{U} \phi_2, X_1 \dotplus X_2]$ (check that this is well defined). It suffices to show that $\beta \circ (K_{1*} \times K_{2*})$ and $(K_{1*} \times K_{2*}) \circ \beta$ are both the identity (then β is a ring isomorphism). First let $[T,\phi,x]$ $\in A_F(S_1 \dot{U} S_2)$. Then $\beta \circ (K_{1*} \times K_{2*})([T,\phi,x])$ $= \beta(Tx_{S_1\dot{U}S_2}S_1, \pi_{S_1}, \pi_T^0(x)], [Tx_{S_1\dot{U}S_2}S_2, \pi_{S_2}, \tilde{\pi}_T^0(x)])$

$= [(T\times_{S_1\dot{\cup} S_2} S_1) \mathbin{\dot{\cup}} (T\times_{S_1\dot{\cup} S_2} S_2), \pi_{S_1} \mathbin{\dot{\cup}} \pi_{S_2}, \pi_T^0(x) \dotplus \tilde{\pi}_T^0(x)]$, where the following pullback diagrams explain our notation.

$$\begin{tikzcd} T\times_{S_1\dot{\cup} S_2} S_1 \arrow[r, "\pi_{S_1}"] \arrow[d, "\pi_T"'] & S_1 \arrow[d, "K_1"] \\ T \arrow[r, "\phi"'] & S_1 \mathbin{\dot{\cup}} S_2 \end{tikzcd} \qquad \begin{tikzcd} T\times_{S_1\dot{\cup} S_2} S_2 \arrow[r, "\pi_{S_2}"] \arrow[d, "\tilde{\pi}_T"'] & S_2 \arrow[d, "K_2"] \\ T \arrow[r, "\phi"'] & S_1 \mathbin{\dot{\cup}} S_2 \end{tikzcd}$$

Define $\gamma: (T\times_{S_1\dot{\cup} S_2} S_1) \mathbin{\dot{\cup}} (T\times_{S_1\dot{\cup} S_2} S_2) \to T$ to be $\gamma = \pi_T \mathbin{\dot{\cup}} \tilde{\pi}_T$. Then γ is a G-isomorphism such that $\phi\gamma = \pi_{S_1} \mathbin{\dot{\cup}} \pi_{S_2}$. We claim that $\gamma^0(x) = \pi_T^0(x) + \tilde{\pi}_T^0(x)$. Let $\lambda_i: T\times_{S_1\dot{\cup} S_2} S_i \to (T\times_{S_1\dot{\cup} S_2} S_1) \mathbin{\dot{\cup}} (T\times_{S_1\dot{\cup} S_2} S_2)$ be inclusion. By the additivity of F, and symmetry, it suffices to show that $\lambda_1^0\gamma^0(x) = \pi_T^0(x)$. This equation follows since $\gamma\lambda_1 = \pi_T$. Thus $\gamma: ((T\times_{S_1\dot{\cup} S_2} S_1) \mathbin{\dot{\cup}} (T\times_{S_1\dot{\cup} S_2} S_2), \pi_{S_1} \mathbin{\dot{\cup}} \pi_{S_2}, \pi_T^0(x) \dotplus \tilde{\pi}_T^0(x)) \to (T, \phi, x)$ is an isomorphism, so that $\beta \circ (K_{1*} \times K_{2*})$ is the identity on $A_F(S_1 \mathbin{\dot{\cup}} S_2)$.

Conversely, let $([T_1, \phi_1, x_1], [T_2, \phi_2, x_2]) \in A_F(S_1) \times A_F(S_2)$. Then as easy computation shows $(K_{1*} \times K_{2*}) \circ \beta([T_1, \phi_1, x_1][T_2, \phi_2, x_2]) = ([(T_1 \mathbin{\dot{\cup}} T_2)\times_{S_1\dot{\cup} S_2} S_1, \pi_{S_1}, \pi^0_{T_1\dot{\cup} T_2}(X_1 + X_2)], [(T_1 \mathbin{\dot{\cup}} T_2)\times_{S_1\dot{\cup} S_2} S_2, \pi_{S_2}, \tilde{\pi}^0_{T_1\dot{\cup} T_2}(X_1 \dotplus X_2)])$ (the reader can deduce our notation). By symmetry, it suffices to show that $(T_1, \phi_1, x_1) \cong ((T_1 \mathbin{\dot{\cup}} T_2)\times_{S_1\dot{\cup} S_2} S_1, \pi_{S_1},$

$\pi^0_{T_1\dot{\cup}T_2}(X_1 \dot{+} X_2))$. Define $\gamma:(T_1 \dot{\cup} T_2)x_{S_1\dot{\cup}S_2}S_1 \to T_1$ by $\gamma(t,s) = t$. Note that if $(t,s) \in (T_1 \dot{\cup} T_2)x_{S_1\dot{\cup}S_2}S_1$, then $s = K_1(s) = (\phi_1 \dot{\cup} \phi_2)(t) \in S_1$. Thus $t \in T_1$ and $s = \phi_1(t)$. It follows that γ is an isomorphism. Plainly, $\phi_1\gamma = \pi_{S_1}$, so finally we must check that $\gamma^0(X_1) = \pi^0_{T_1\dot{\cup}T_2}(X_1 \dot{+} X_2)$. If $\lambda:T_1 \to T_1 \dot{\cup} T_2$ is inclusion, then $\pi_{T_1\dot{\cup}T_2} = \lambda\gamma$. Thus $\pi^0_{T_1\dot{\cup}T_2}(X_1 \dot{+} X_2) = \gamma^0\lambda^0(X_1 \dot{+} X_2) = \gamma^0(X_1)$. Therefore, γ is the required isomorphism.

Axiom 2.7(a). Let S be a G-set, and let G/G denote the one-point G-set. Define $I_S:S \to S$ in the only possible way, and let 1_S be the unit of $F(S)$. Then it is easy to check that $[S,I_S,1_S] = 1_{A_F(S)}$.

Axiom 2.7(b). This is shown in 3.5.

Axiom 2.7(c). Let $\alpha:S \to T$ be a G-map. Let $[V,\phi,x] \in A_F(S)$ and $[W,\psi,y] \in A_F(T)$. We must show that $\alpha^*(\alpha_*([W,\psi,y]) \cdot [V,\phi,x]) = [W,\psi,y] \cdot \alpha^*([V,\phi,x])$. After applying the definitions of α^* and α_* it is enough to show that $((Wx_TS)x_SV,\alpha \circ (\pi_Sx_S\phi),(\pi^0_{Wx_TS}\pi^0_W(y)) \cdot (\pi^0_V(x)))$ $\cong (Wx_TV,\psi x_T(\alpha\phi),\tilde{\pi}^0_W(y) \cdot \tilde{\pi}^0_V(x))$, where the following pullback diagrams explain our notation.

$$\begin{tikzcd} Wx_TS \arrow[r, "\pi_S"] \arrow[d, "\pi_W"'] & S \arrow[d, "\alpha"] \\ W \arrow[r, "\psi"'] & T \end{tikzcd} \qquad \begin{tikzcd} (Wx_TS)x_SV \arrow[r, "\pi_V"] \arrow[d, "\pi_{Wx_TS}"'] & V \arrow[d, "\phi"] \\ Wx_TS \arrow[r, "\pi_S"'] & S \end{tikzcd} \qquad \begin{tikzcd} Wx_TV \arrow[r, "\tilde{\pi}_V"] \arrow[d, "\tilde{\pi}_W"'] & V \arrow[d, "\alpha\phi"] \\ W \arrow[r, "\psi"'] & T \end{tikzcd}$$

Define $\gamma:(Wx_TS)x_SV \to Wx_TV$ by $\gamma((w,s),v) = (w,v)$. Then γ is a G-isomorphism such that $\alpha \circ (\pi_Sx_S\phi) = (\psi x_T(\alpha\phi)) \circ \gamma$ (as one checks). Moreover, since $\pi_W\pi_{Wx_TS} = \tilde{\pi}_W\gamma$ and $\pi_V = \tilde{\pi}_V\gamma$, it follows that $\gamma^0(\tilde{\pi}_W^0(y) \cdot \tilde{\pi}_V^0(x)) = (\tilde{\pi}_W\gamma)^0(x) \cdot (\tilde{\pi}_V\gamma)^0(y) = (\pi^0_{Wx_TS}\pi^0_W(x)) \cdot (\pi_V^0(y))$. Thus γ gives us the required isomorphism. □

<u>A Basis for</u> $A_F(G)$

We introduce some notational conveniences. If $H \leq G$, then G/H denotes the transitive G-set of left cosets modulo H. We will denote $A_F(G/H)$ by $A_F(H)$. In particular, if $H = G$, then for any non-empty G-set T, there is exactly one G-map $\eta_T:T \to G/G$. Thus we abbreviate the category $(G,G/G,F)$ to (G,F), the element $[T,\eta_T,x]$ of $A_F(G)$ to $[T,x]$, and the object (T,η_T,x) of (G,F) to (T,x). Then isomorphism in (G,F) of objects (T,x) and (V,y) is equivalent with the existence of a G-isomorphism $\beta:T \to V$ with $\beta^0(y) = x$.

For any G-set T, let $W_T = Aut_G(T)$. Especially, if $a \in P$, we shall abbreviate W_{S_a} to $W_a = Aut_G(S_a)$.

We use W_T to define an equivalence relation $\sim_T$ on $F(T)$, namely, we say $x \sim_T y$ if and only if there exists $\alpha \in W_T$ with $\alpha^0(x) = y$, x, $y \in F(T)$. For $a \in P$ we shall let $x \sim_a y$ denote $x \sim_{S_a} y$. The following lemma is a direct consequence of these definitions and Corollary 3.4.

Lemma 3.7. Let T be a G-set, and let x, $y \in F(T)$. Then $x \sim_T y$ if and only if $[T,x] = [T,y]$ in $A_F(G)$.

Let $\underline{\gamma} = \underline{\gamma}(G) = \{S_a : a \in P\}$. By 2.1(a), $\underline{\gamma}$ is a complete set of representatives of isomorphism classes of transitive G-sets. For each $a \in P$, choose a set $R_a \subseteq F(S_a)$ of equivalence class representatives under $\sim_a$. The following propostion may be viewed as the uniqueness statement in Wedderburn's theorem.

Proposition 3.8. Fix $a \in P$ and suppose that $\sum_{i=1}^{m} [S_a, x_i] = \sum_{i=1}^{n} [S_a, y_i]$ for some x_i, $y_i \in R_a$. Then $m = n$, and there is a permutation π of $\{1, \ldots, n\}$ such that $x_i = y_{\pi(i)}$, all i.

Proof. By 3.4, $(\dot{\bigcup}_{i=1}^{m} S_a, x_1 \dotplus \ldots \dotplus x_m) \cong (\dot{\bigcup}_{i=1}^{n} S_a, y_1 \dotplus \ldots \dotplus y_n)$, in particular, $\dot{\bigcup}_{i=1}^{m} S_a \cong \dot{\bigcup}_{i=1}^{n} S_a$, so $m = n$. For notational ease, we set $S_a^i = S_a$, $1 \leq i \leq n$. Choose an isomorphism $\alpha : \dot{\bigcup}_{i=1}^{n} S_a^i \to \dot{\bigcup}_{i=1}^{n} S_a^i$ with $\alpha^0(y_1 \dotplus \ldots \dotplus y_n) = x_1 \dotplus \ldots \dotplus x_n$. For each i, $\alpha(S_a^i)$ is a transitive subset of

$\dot{\bigcup}_{j=1}^{n} S_a^j$, so there is an index $\pi(i)$ with $\alpha(S_a^i) = S_a^{\pi(i)}$. This defines π. Since α is an isomorphism, π is a permutation of $\{1, \ldots, n\}$. For each i, let $K_i : S_a^i \to \dot{\bigcup}_{j=1}^{n} S_a^j$ be inclusion, and let $\alpha_i = \alpha|_{S_a^i} : S_a^i \to S_a^{\pi(i)}$. Plainly, $\alpha K_i = K_{\pi(i)}\alpha_i$. Thus $\alpha_i^0(y_{\pi(i)}) = \alpha_i^0 K_{\pi(i)}^0 (y_1 \dot{+} \cdots \dot{+} y_n)$ $= K_i^0 \alpha^0 (y_1 \dot{+} \cdots \dot{+} y_n) = K_i^0(x_1 \dot{+} \cdots \dot{+} x_n) = x_i$. Since $\alpha_i : S_a^i \to S_a^{\pi(i)}$ is an isomorphism, and $S_a^i = S_a^{\pi(i)} = S_a$, it follows from the fact that x_i, $y_{\pi(i)} \in R_a$ that $x_i = y_{\pi(i)}$, all i. □

<u>Theorem 3.9</u>. Let $F : G \to AM$ be an additive contravariant functor. Define $B_F = \{[S_a, x] : a \in P(G),\ x \in R_a\}$. Then B_F is a $\mathbb{Z}$-basis of $A_F(G)$.

<u>Proof</u>. Let $[T,y] \in A_F(G)$. Write $T = \dot{\bigcup}_{i=1}^{n} T_i$, with each T_i a transitive G-set. By additivity of F, we may find elements $y_i \in F(T_i)$ with $[T,y] = \sum_{i=1}^{n} [T_i, y_i]$. For each i, choose $a_i \in P$ and an isomorphism $\alpha_i : S_{a_i} \to T_i$. Then, for each i, there is a unique $x_i \in R_{a_i}$ with $\alpha_i^0(y_i) \sim_{a_i} x_i$. Thus $(T_i, y_i) \cong (S_{a_i}, \alpha_i^0(y_i)) \cong (S_{a_i}, x_i)$, so that $[T,y]$ $= \sum_{i=1}^{n} [S_a, x_i]$, and B_F spans.

For independence, first suppose there is a dependence relation $\sum_{i=1}^{n} c_i [S_a, x_i] = 0$ for some fixed $a \in P$, where

$x_i \neq x_j$ if $i \neq j$, and c_i is non-zero, all i. Then by Proposition 3.8, the equality $\sum_{c_i>0} c_i[S_a,x_i] = \sum_{c_j<0} (-c_j)[S_a,x_j]$ yields $x_i = x_j$ for some $i \neq j$, a contradiction. In general, if there is a dependence relation $\sum_{a\in P} \sum_{x\in R_a} c_{a,x}[S_a,x] = 0$, then since the S_a are pairwise non-isomorphic, Corrollary 3.4 yields $\sum_{x\in R_a} c_{a,x}[S_a,x] = 0$, for each $a \in P$. By the above argument, $c_{a,x} = 0$ for all $a \in P$, $x \in R_a$. □

Finally, let us consider the case when the relation $\sim$ is trivial.

Definition 3.10. Let T be a G-set. An element $x \in F(T)$ is normal if given any $\alpha \in W_T = \mathrm{Aut}_G(T)$, we have $\alpha^0(x) = x$. Let $F_N(T)$ denote the set of all normal elements of $F(T)$. A G-set T is normal over F if $F_N(T) = F(T)$. If every G-set T is normal over F then F is called normal.

We collect some facts about normality.

Proposition 3.11. Let $F:\hat{G} \to AM$ be an additive contravariant functor, and let T be any G-set.

(a) $F_N(T)$ is a subgroup of $F(T)$. In fact, if we let $W_T^0 = \{\alpha^0 : \alpha \in W_T\} \subseteq \mathrm{Aut}(F(T))$, then $F_N(T)$ is the fixed subgroup of $F(T)$ under the action of W_T^0.

(b) If $\eta_T : T \to G/G$ denotes the canonical map, then image $(\eta_T^0) \subseteq F_N(T)$.

(c) If S_a is normal over F, then $R_a = F(S_a)$. In particular, if F is normal, then $B_F = \{[S_a, x] : a \in P, x \in F(S_a)\}$.

The proofs of these statements are trivialities. Under the assumption of normality for the functor F, the Green-functor A_F is especially computable. Indeed, its theory resembles that of the Burnside ring functor A. It will be the topic of Chapter 6 to describe some of these connections.

CHAPTER 4

FUNCTORIAL PROPERTIES

Fixed throughout this chapter is a finite group G. We shall denote by AM^G the category of additive contravariant functors $F:\hat{G} \to AM$, with natural transformations as morphisms, and by GF^G the category of Green-functors $M:\hat{G} \to AB$. Given $M \in GF^G$, it follows that $M_* \in AM^G$, where for a G-set S, $M_*(S)$ is the multiplicative monoid of $M(S)$. By axioms 2.6(b), 2.7(a) and 2.7(b), we obtain the forgetful functor $U:GF^G \to AM^G$ given by $U(M) = M_*$. By general existence theorems, a left adjoint must exist for U. The purpose of our present discussion is to show that the correspondence $F \to A_F$, from AM^G to GF^G, defines such an adjoint. We must first establish that this correspondence defines a functor.

<u>Proposition 4.1</u>. Let F_1, $F_2 \in AM^G$, and let $\gamma:F_1 \to F_2$ be a natural transformation. Then there is an induced natural transformation of Green-functors $\hat{\gamma}:A_{F_1} \to A_{F_2}$, such that for all $S \in \hat{G}$, $[T,\phi,x] \in A_{F_1}(S)$, $\hat{\gamma}_S([T,\phi,x]) = [T,\phi,\gamma_T(x)] \in A_{F_2}(S)$.

Proof. Let $S \in \hat{G}$. Define $\lambda_S:(G,S,F_1) \to A_{F_2}(S)$ by $\lambda_S(T,\phi,x) = [T,\phi,\gamma_T(x)]$. We must first check that λ_S respects isomorphism, $\oplus$, and x_S. Let $(T_i,\phi_i,x_i) \in (G,S,F_1)$, $i = 1, 2$.

i) Suppose $\alpha:(T_1,\phi_1,x_1) \to (T_2,\phi_2,x_2)$ is an isomorphism, so that $F_1(\alpha)(x_2) = x_1$ and $\phi_1 = \phi_2\alpha$. Since γ is a natural transformation, $F_2(\alpha)\gamma_{T_2}(x_2) = \gamma_{T_1}F_1(\alpha)(x_2) = \gamma_{T_1}(x_1)$. It follows that $\alpha:(T_1,\phi_1,\gamma_{T_1}(x_1)) \to (T_2,\phi_2,\gamma_{T_2}(x_2))$ is an isomorphism in (G,S,F_2). Thus, by Corollary 3.4, $\lambda_S(T_1,\phi_1,x_1) = [T_1,\phi_1,\gamma_{T_1}(x_1)] = [T_2,\phi_2,\gamma_{T_2}(x_2)] = \lambda_S(T_2,\phi_2,x_2)$.

ii) λ_S respects $\oplus$. Need $\lambda_S(T_1,\phi_1,x_1) + \lambda_S(T_2,\phi_2,x_2) = \lambda_S((T_1,\phi_1,x_1) \oplus (T_2,\phi_2,x_2))$, that is, $[T_1 \dot{\cup} T_2,\phi_1 \dot{\cup} \phi_2,\gamma_{T_1}(x_1) \dot{+} \gamma_{T_2}(x_2)] = [T_1 \dot{\cup} T_2,\phi_1 \dot{\cup} \phi_2, \gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2)]$. It suffices to show that $\gamma_{T_1}(x_1) \dot{+} \gamma_{T_2}(x_2) = \gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2)$ in $F_2(T_1 \dot{\cup} T_2)$. By naturality, $\gamma_{T_i}F_1(K_i) = F_2(K_i)\gamma_{T_1\dot{\cup}T_2}:F_1(T_1 \dot{\cup} T_2) \to F_2(T_i)$, $i = 1, 2$. Thus, $F_2(K_i)\gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2) = \gamma_{T_i}F_1(K_i)(x_1 \dot{+} x_2) = \gamma_{T_i}(x_i)$. By additivity of F_2, this shows $\gamma_{T_1}(x_1) \dot{+} \gamma_{T_2}(x_2) = \gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2)$, as needed.

iii) λ_S respects x_S. Computing, as above, we must show that $[T_1 x_S T_2,\phi_1 x_S \phi_2,\gamma_{T_1 x_S T_2}(F_1(\pi_1)(x_1) \cdot F_1(\pi_1)(x_2))]$

$= [T_1 x_S T_2, \phi_1 x_S \phi_2, F_2(\pi_1)(\gamma_{T_1}(x_1)) \cdot F_2(\gamma_2)(\pi_{T_2}(x_2))]$, so it suffices to show $\gamma_{T_1 x_S T_2}(F_1(\pi_1)(x_1) \cdot F_2(\pi_2)(x_2))$ $= F_2(\pi_1)(\gamma_{T_1}(x_1)) \cdot F_2(\pi_2)(\gamma_{T_2}(x_2))$ in $F_2(T_1 x_S T_2)$. This follows immediately, since by naturality of γ, $\gamma_{T_1 x_S T_2} F_1(\pi_i) = F_2(\pi_i)\gamma_{T_i}$, $i = 1, 2$.

It follows that there is an induced ring homomorphism $\hat{\gamma}_S : A_{F_1}(S) \to A_{F_2}(S)$ satisfying $\hat{\gamma}_S([T,\phi,x]) = [T,\phi,\gamma_T(x)]$. We now show $\hat{\gamma} = \{\hat{\gamma}_S : S \in \hat{G}\}$ is a natural transformation of Green-functors $A_{F_1} \to A_{F_2}$. Let $\alpha : S \to T$ be a G-map. We must show that $\hat{\gamma}_T A^*_{F_1}(\alpha) = A^*_{F_2}(\alpha)\hat{\gamma}_S : A_{F_1}(S) \to A_{F_2}(T)$, and that $\hat{\gamma}_S A_{F_1 *}(\alpha) = A_{F_2 *}(\alpha)\hat{\gamma}_T : A_{F_1}(T) \to A_{F_2}(S)$.

If $[V,\phi,x] \in A_{F_1}(S)$, then $\hat{\gamma}_T A^*_{F_1}(\alpha)([V,\phi,x])$ $= \hat{\gamma}_T([V,\alpha\phi,x]) = [V,\alpha\phi,\gamma_V(x)] = A^*_{F_2}(\alpha)([V,\phi,\gamma_V(x)])$ $= A^*_{F_2}(\alpha)\hat{\gamma}_S([V,\phi,x])$.

Conversely, if $[W,\psi,y] \in A_{F_1}(T)$, then $\hat{\gamma}_S A_{F_1 *}(\alpha)([W,\psi,y]) = \hat{\gamma}_S([Wx_T S, \pi_S, F_1(\pi_W)(y)]) = [Wx_T S, \pi_S, \gamma_{Wx_T S} F_1(\pi_W)(y)]$, whereas, $A_{F_2 *}(\alpha)\hat{\gamma}_T([W,\psi,y])$ $= A_{F_2 *}(\alpha)([W,\psi,\gamma_W(y)]) = [Wx_T S, \pi_S, F_2(\pi_W)\gamma_W(y)]$. By naturality of γ, $\gamma_{Wx_T S} F_1(\pi_W)(y) = F_2(\pi_W)\gamma_W(y)$. □

Corollary 4.2. The correspondences $F \to A_F$, $\gamma \to \hat{\gamma}$ define a covariant functor from AM^G to GF^G.

Conversely, we have the following.

Proposition 4.3. Let $F \in AM^G$, and $M \in GF^G$. Given any natural transformation $\gamma: F \to U(M)$, the prescripition $\tilde{\gamma}_S([T,\phi,x]) = M^*(\phi)\gamma_T(x): A_F(S) \to M(S)$ defines a natural transformation of Green-functors $\tilde{\gamma}: A_F \to M$.

Proof. Fix $S \in \hat{G}$. Define $\lambda_S: (G,S,F) \to M(S)$ by $\lambda_S(T,\phi,x) = \phi^*\gamma_T(x)$, where $\phi^* = M^*(\phi): M(T) \to M(S)$. As usual, let $(T_i,\phi_i,x_i) \in (G,S,F)$, $i = 1, 2$.

i) λ_S respects isomorphism. Suppose $\alpha: (T_1,\phi_1,x_1) \to (T_2,\phi_2,x_2)$ is an isomorphism, so that $\alpha^0(x_2) = x_1$ and $\phi_1 = \phi_2\alpha$. By Frobenius reciprocity (2.7(c)), $\lambda_S(T_1,\phi_1,x_1) = \phi_1^*\gamma_{T_1}(x_1) = \phi_2^*\alpha^*\gamma_{T_1}(\alpha^0(x_2)) = \phi_2^*\alpha^*\alpha_*\gamma_{T_2}(x_2) = \phi_2^*(\gamma_{T_2}(x_2) \cdot \alpha^*(1_{M(T_1)})) = \phi_2^*\gamma_{T_2}(x_2) = \lambda_S(T_2,\phi_2,x_2)$.

ii) λ_S is additive. By Frobenius reciprocity, naturality of γ, and Proposition 2.10, we have

$$\lambda_S(T_1,\phi_1,x_1) + \lambda_S(T_2,\phi_2,x_2) = \phi_1^*\gamma_{T_1}(x_1) + \phi_2^*\gamma_{T_2}(x_2)$$

$$= (\phi_1 \dot{\cup} \phi_2 \circ K_1)^*\gamma_{T_1}(K_1^0(x_1 \dot{+} x_2))$$

$$+ (\phi_1 \dot{\cup} \phi_2 \circ K_2)^*\gamma_{T_2}(K_2^0(x_1 \dot{+} x_2))$$

$$= (\phi_1 \dot{\cup} \phi_2)^*K_1^*K_{1*}\gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2)$$

$$+ (\phi_1 \dot{\cup} \phi_2)^*K_2^*K_{2*}\gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2)$$

$$= (\phi_1 \dot{\cup} \phi_2)^*(\gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2) \cdot K_1^*(1_{M(T_1)}))$$

$$+ (\phi_1 \dot{\cup} \phi_2)^*(\gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2) \cdot K_2^*(1_{M(T_2)}))$$

$$= (\phi_1 \dot{\cup} \phi_2)^*(\gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2)((K_1^*K_{1*} + K_2^*K_{2*})(1_{M(T_1\dot{\cup}T_2)})))$$

$$= (\phi_1 \dot{\cup} \phi_2)^*\gamma_{T_1\dot{\cup}T_2}(x_1 \dot{+} x_2) = \lambda_S((T_1,\phi_1,x_1) \ominus (T_2,\phi_2,x_2)).$$

iii) λ_S respects x_S. Using Frobenius reciprocity, and 2.6(a) we have

$$\lambda_S((T_1,\phi_1,x_1)x_S(T_2,\phi_2,x_2)) = \lambda_S((T_1x_ST_2,\phi_1x_S\phi_2,\pi_1^0(x_1) \cdot \pi_2^0(x_2)))$$

$$= (\phi_1x_S\phi_2)^*\gamma_{T_1x_ST_2}(\pi_1^0(x_1) \cdot \pi_2^0(x_2))$$

$$= (\phi_1x_S\phi_2)^*(\gamma_{T_1x_ST_2}(\pi_1^0(x_1)) \cdot \gamma_{T_1x_ST_2}(\pi_2^0(x_2)))$$

$$= \phi_1^*\pi_1^*(\pi_{1*}\gamma_{T_1}(x_1) \cdot \pi_{2*}\gamma_{T_2}(x_2)) = \phi_1^*(\gamma_{T_1}(x_1) \cdot \pi_1^*\pi_{2*}\gamma_{T_2}(x_2))$$

$$= \phi_1^*(\gamma_{T_1}(x_1) \cdot \phi_{1*}\phi_2^*\gamma_{T_2}(x_2)) = \phi_2^*\gamma_{T_2}(x_2) \cdot \phi_1^*\gamma_{T_1}(x_1)$$

$$= \lambda_S(T_1,\phi_1,x_1) \cdot \lambda_S(T_2,\phi_2,x_2).$$

It follows that λ_S induces a ring homomorphism $\tilde{\gamma}_S:A_F(S) \to M(S)$ satisfying $\tilde{\gamma}_S([T,\phi,x]) = M^*(\phi)\gamma_T(x)$. To see that $\tilde{\gamma} = \{\tilde{\gamma}_S:S \in \hat{G}\}:A_F \to M$ is a natural transformation of Green-functors, let $\alpha:S \to T$ be a G-map. We must show that $\tilde{\gamma}_T A_F^*(\alpha) = M^*(\alpha)\tilde{\gamma}_S:A_F(S) \to M(T)$, and that $\tilde{\gamma}_S A_{F*}(\alpha) = M_*(\alpha)\tilde{\gamma}_T:A_F(T) \to M(S)$.

If $[V,\phi,x] \in A_F(S)$, then $\tilde{\gamma}_T A_F^*(\alpha)([V,\phi,x])$
$= \tilde{\gamma}_T([V,\alpha\phi,x]) = M^*(\alpha\phi)\gamma_V(x) = M^*(\alpha)M^*(\phi)\gamma_V(x)$
$= M^*(\alpha)\tilde{\gamma}_S([V,\phi,x])$.

Conversely, if $[W,\psi,y] \in A_F(T)$, then
$\tilde{\gamma}_S A_{F*}(\alpha)([W,\psi,y]) = \tilde{\gamma}_S([Wx_TS,\pi_S\pi_W^0(y)]) = \pi_S^*\gamma_{Wx_TS}(\pi_W^0(y))$
$= \pi_S^*\pi_{W*}\gamma_W(y) = \alpha_*\phi^*\gamma_W(y) = M_*(\alpha)\tilde{\gamma}_T([W,\psi,y])$, using 2.6(a).

We can now prove the main theorem of this chapter.

<u>Theorem 4.4</u>. The functor $F \to A_F$ from AM^G to GF^G is the left adjoint of the forgetful functor $U:GF^G \to AM^G$.

<u>Proof</u>. Fix $F \in AM^G$, $M \in GF^G$. We must establish a natural bijection $\mathrm{Nat}(A_F,M) \leftrightarrow \mathrm{Nat}(F,UM)$. Define $\Phi:\mathrm{Nat}(A_F,M) \to \mathrm{Nat}(F,UM)$ by $\Phi(\gamma)_S(x) = \gamma_S([S,1_S,x])$, and $\Psi:\mathrm{Nat}(F,UM) \to \mathrm{Nat}(A_F,M)$ by $\Psi(\gamma) = \tilde{\gamma}$ (which is well defined by 4.3). We now show that Φ and Ψ are inverse bijections.

If $\gamma \in \mathrm{Nat}(A_F,M)$, $S \in \hat{G}$, and $[T,\phi,x] \in A_F(S)$, then $(\Psi\Phi(\gamma))_S([T,\phi,x]) = \Phi(\tilde{\gamma})_S([T,\phi,x]) = M^*(\phi)\Phi(\gamma)_T(x)$
$= M^*(\phi)\gamma_T([T,1_T,x]) = \gamma_S A_F^*(\phi)([T,1_T,x]) = \gamma_S([T,\phi,x])$. Hence $\Psi\Phi = 1$.

If $\gamma \in \mathrm{Nat}(F,UM)$, $S \in \hat{G}$, and $x \in F(S)$, then $(\Phi\Psi(\gamma))_S(x) = \Psi(\gamma)_S([S,1_S,x]) = \tilde{\gamma}_S([S,1_S,x]) = M^*(1_S)\gamma_S(x)$
$= \gamma_S(x)$. Therefore $\Phi\Psi = 1$. All that remains is to show naturality in F and M.

For the 'F' variable, let $\gamma:F_1 \to F_2$ be a natural transformation in AM^G, and let $\Psi_i:\mathrm{Nat}(F_i,UM) \to \mathrm{Nat}(A_{F_i}, M)$

be the function given above, $i = 1, 2$. We must show that for any $\Theta \in \mathrm{Nat}(F_2, UM)$, we have $\Psi_1(\Theta\gamma) = \Psi_2(\Theta)\hat{\gamma}: A_{F_1} \to M$. Let $S \in \hat{G}$ and $[T,\phi,x] \in A_{F_1}(S)$. Then $\Psi_1(\Theta\gamma)_S([T,\phi,x])$ $= (\widetilde{\Theta\gamma})_S([T,\phi,x]) = M^*(\phi)(\Theta\gamma)_T(x) = M^*(\phi)\Theta_T\gamma_T(x)$ $= \Psi_2(\Theta)_S([T,\phi,\gamma_T(x)]) = \Psi_2(\Theta)_S\hat{\gamma}_S([T,\phi,x])$.

For the 'M' variable, fix $F \in AM^G$, and let $\gamma: M_1 \to M_2$ be a natural transformation of Green-functors. We must show that for any $\Theta \in \mathrm{Nat}(F, UM_1)$, we have $\gamma\Psi_1(\Theta)$ $= \Psi_2(\gamma\Theta): A_F \to M_2$. Let $S \in \hat{G}$ and $[T,\phi,x] \in A_F(S)$. Then $(\gamma\Psi_1(\Theta))_S([T,\phi,x]) = \gamma_S\Psi_1(\Theta)_S([T,\phi,x]) = \gamma_S\tilde{\Theta}_S([T,\phi,x])$ $= \gamma_S M_1^*(\phi)\Theta_T(x) = M_2^*(\phi)\gamma_T\Theta_T(x) = M_2^*(\phi)(\gamma\Theta)_T(x)$ $= \Psi_2(\gamma\Theta)_S([T,\phi,x])$. □

Of course, if we let $M = A_F$, then adjointness implies that the identity transformation $1_{A_F} \in \mathrm{Nat}(A_F, A_F)$ determines a universal arrow $\Phi(1_{A_F}): F \to UA_F$ (MacLane 1971, pp. 77-84). Explicitly, we have $\Phi(1_{A_F})_S(x) = [S, 1_S, x]$, all $S \in \hat{G}$, $x \in F(S)$, and the universality may be rephrased thus:

<u>Corollary 4.5</u>. Let $F \in AM^G$, and $\Phi(1_{A_F}): F \to UA_F$ be the natural transformation given above. Then, given any Green-functor M, and natural transformation $\gamma: F \to UM$, there is a natural transformation of Green-functors $\tilde{\gamma}: A_F \to M$ (given as in 4.3) such that $\gamma = \tilde{\gamma}\Phi(1_{A_F})$.

Now the functor $I:\hat{G} \to AM$, which associates to each G-set the monoid consisting of the identity alone, is both an initial and final object in AM^G. For each $F \in AM^G$, we let $\alpha_F:I \to F$ and $\zeta_F:F \to I$ be the canonical natural transformations. Since $\zeta_F \alpha_F:I \to I$ is the identity, it follows that for each G-set S, $\hat{\alpha}_{F,S}:A_I(S) \to A_F(S)$ embeds $A_I(S)$ as a direct summand of $A_F(S)$, and that $\hat{\zeta}_{F,S}:A_F(S) \to A_I(S)$ is surjective. In particular, the correspondence $(T,\phi) \to [T,\phi,1]$ is an isomorphism $A(S) \to A_I(S)$, where $A(S)$ the Burnside ring of G-sets over S (Dress 1971, pp. 54-61), and thus we may (and do) identify $A(S)$ with a subring of $A_F(S)$. Explicitly, $A(S) \cong \text{image } (\hat{\alpha}_{F,S} \circ \hat{\zeta}_{F,S}) \subseteq A_F(S)$, any $F \in AM^G$. This observation will be useful later when we shall exploit the known properties of $A(S)$ in determining those of $A_F(S)$. For example, using the fact that $\hat{\alpha}_F$ and $\hat{\zeta}_F$ are natural transformations of Green-functors, together with the fact that $A \cong A_I$ is an initial object in GF^G (Dress 1971, p. 79), we obtain

<u>Corollary 4.6</u>. For any $F \in AM^G$, A_F is an initial object in the category of Green-functors: $\hat{G} \to AB$.

Finally, we can compute the defect basis of A_F. Indeed, since the defect basis of the Burnside ring functor is the set of all subgroups of G, the following corollary is obtained.

Corollary 4.7. For any $F \in AM^G$, the defect basis of A_F is the set of all subgroups of G.

Proof. This follows directly from Dress (1971, p. 87), and the existence of $\hat{\alpha}_F$ and $\hat{\zeta}_F$. $\square$

CHAPTER 5

STRUCTURE THEORY

Fixed in this chapter are a finite group G, and a functor $F \in AM^G$. By Theorem 3.9, $A_F(G)$ is torsion free (as an abelian group), and thus it embeds faithfully in the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} A_F(G)$. For simplicity we shall denote $\mathbb{Q} \otimes_{\mathbb{Z}} A_F(G)$ by $\mathbb{Q}A_F(G)$, and consider its elements to be rational multiples of elements of $A_F(G)$. The principal aim of this chapter is the explicit computation of $\mathbb{Q}A_F(G)$. In the next chapter we will use this characterization to examine the prime ideal structure of $A_F(G)$ when F is normal.

<u>The Structure of</u> $\mathbb{Q}A_F(G)$

As discussed at the end of Chapter 4, $A(G) \cong A_I(G)$, and we may identify $A(G)$ with the subring of $A_F(G)$ consisting of the elements $\{[S,1] - [T,1]:S,T \in \hat{G}\}$. In particular, from Chapter 2, $\mathbb{Q}A(G)$ has primitive idempotents $\{e_a:a \in P\}$, where $e_a = \sum_{b\in P} \lambda_{b,a}[S_b,1]$, and multiplication in $\mathbb{Q}A(G)$ satisfies $[S_a,1][S_b,1] = \sum_{c\in P} V_{a,b,c}[S_c,1]$.

<u>Lemma 5.1</u>. Let a, $b \in P$ and $x \in F(S_a)$. Then for some $r \geq 0$, $[S_a,x][S_b,1] = V_{a,b,a}[S_a,x] + \sum_{j=1}^{r} [S_{a_j},x_j]$, where $a_j < a$ and $x_j \in F(S_{a_j})$, $1 \leq j \leq r$.

<u>Proof</u>. Set $n = V_{a,b,a}$. If $a \not\leq b$, then $n = 0$ and the result is clear. Assume $a \leq b$, and set $S_a^i = S_a$, $1 \leq i \leq n$ (possibly $n = 0$, but this gives no trouble). Then $S_a \times S_b \cong S_a^1 \dot{\cup} \ldots \dot{\cup} S_a^n \dot{\cup} \dot{\bigcup}_{j=1}^{r} S_{a_j} = S$, where $a_j < a$, $1 \leq j \leq r$ (by 2.2(c)). Let $\alpha: S \to S_a \times S_b$ be this isomorphism, and let $K_i: S_a^i \to S$, $\ell_j: S_{a_j} \to S$ be the canonical injections. Let $\pi: S_a \times S_b \to S_a$ be the projection map. Since each composite $\pi\alpha K_i: S_a^i = S_a \to S_a$ is a G-map, it must be an automorphism, by the transitivity of S_a. By Lemma 3.7, $[S_a, x] = [S_a^i, (\pi\alpha K_i)^0(x)]$ $1 \leq i \leq n$. Set $x_j = (\pi\alpha\ell_j)^0(x) \in F(S_{a_j})$, $1 \leq j \leq r$. By the additivity of F, and the above comments,

$$\begin{aligned}
[S_a, x][S_b, 1] &= [S_a \times S_b, \pi^0(x)] \\
&= [S, \alpha^0\pi^0(x)] \\
&= \sum_{i=1}^{n} [S_a^i, K_i^0\alpha^0\pi^0(x)] + \sum_{j=1}^{r} [S_{a_j}, \ell_j^0\alpha^0\pi^0(x)] \\
&= \sum_{i=1}^{n} [S_a, x] + \sum_{j=1}^{r} [S_{a_j}, x_j] \\
&= V_{a,b,a}[S_a, x] + \sum_{j=1}^{r} [S_{a_j}, x_j]. \qquad \square
\end{aligned}$$

We now generalize Proposition 2.4.

<u>Proposition 5.2</u>. Let a, $b \in P$ with $b \not\leq a$, and let $x \in F(S_a)$. Then $[S_a,x]e_b = 0$.

<u>Proof</u>. The proof proceeds by induction on $a \in P$ with respect to $\leq$. If $a = 1$, then by 5.1 and 2.3,

$$[S_1,x]e_b = \sum_{c \in P} \lambda_{c,b}[S_1,x][S_c,1]$$

$$= (\sum_{c \in P} \lambda_{c,b}V_{1,c,1})[S_1,x] = 0.$$

Assume $[S_c,y]e_b = 0$ whenever $c < a$ and $y \in F(S_c)$ (thus $b \not\leq c$, since $b \not\leq a$). Then

$$[S_a,x]e_b = [S_a,x]e_b \cdot e_b$$

$$= \sum_{c \in P} \lambda_{c,b}[S_a,x][S_c,1]e_b$$

$$= \sum_{c} \lambda_{c,b}(V_{a,c,a}[S_a,x] + \sum_{j=1}^{r_c} [S_{a_{j,c}},x_{j,c}])e_b$$

$$= (\sum_{c} \lambda_{c,b}V_{a,c,a})[S_a,x]e_b$$

$$+ \sum_{c} \sum_{j=1}^{r_c} \lambda_{c,b}[S_{a_{j,c}},x_{j,c}]e_b.$$

Since each $a_{j,c} < a$, induction implies that all $[S_{a_{j,c}},x_{j,c}]e_b = 0$, and thus, $[S_a,x]e_b$ $= (\sum_{c} \lambda_{c,b}V_{a,c,a})[S_a,x]e_b$. The hypothesis $b \not\leq a$ implies that either $a < b$ or $a \not\leq b$. If $a < b$, then 2.3 implies $\sum_c \lambda_{c,b} V_{a,c,a} = 0$. If $a \not\leq b$, then $a \not\leq c$ for all $c \leq b$, so that $V_{a,c,a} = 0$, all $c \leq b$ by 2.2(c). But if $c \not\leq b$, then $\lambda_{c,b} = 0$ by definition. Hence $\sum_c \lambda_{c,b} V_{a,c,a} = 0$ in this case also. In either case, this implies $[S_a,x]e_b = 0$. □

The next step is the explicit computation of the product $[S_a,x][S_a,y]e_a$, any x, $y \in F(S_a)$. To obtain this, we must recall an isomorphism yielding the decomposition of $S_a \times S_a$ into transitive G-sets. For any $a \in P$, recall that $\mathrm{Aut}_G(S_a) \cong N_G(H_a)/H_a$, in particular $|\mathrm{Aut}_G(S_a)| = V_a$. We just state the following lemma.

Lemma 5.3. Let $a \in P$, and set $S_a^i = S_a$, $1 \leq i \leq V_a$. Say that $\mathrm{Aut}_G(S_a) = \{\sigma_i : 1 \leq i \leq V_a\}$. For each i, define $\alpha_i : S_a^i \to S_a \times S_a$ by $\alpha_i(s) = (s, \sigma_i(s))$. Then there is a (possibly empty) set $\{a_j : 1 \leq j \leq n\} \subseteq P$ with each $a_j < a$, and an isomorphism $\alpha : S_a^1 \dot{\cup} \ldots \dot{\cup} S_a^{V_a} \dot{\cup} \dot{\bigcup}_{j=1}^{n} S_{a_j} \to S_a \times S_a$ such that if $K_i : S_a^i \to S_a^1 \dot{\cup} \ldots \dot{\cup} S_a^{V_a} \dot{\cup} \dot{\bigcup}_{j=1}^{n} S_{a_j}$ is inclusion, then $\alpha_i = \alpha K_i$, all i.

Since F is a functor, there is a natural action of W_S on $F(S)$, for any G-set S, given by $\sigma \cdot x = (\sigma^{-1})^0(x)$, $x \in F(S)$, $\sigma \in W_S$. Contravariance implies $(\sigma\tau) \cdot x = ((\sigma\tau)^{-1})^0(x) = (\tau^{-1}\sigma^{-1})^0(x) = (\sigma^{-1})^0(\tau^{-1})^0(x) = \sigma \cdot (\tau \cdot x)$.

For brevity we denote $\sigma \cdot x$ by x_σ. This action plays a key role in the structure of $\mathbb{Q}A_F(G)$, as illustrated by the following lemma.

<u>Lemma 5.4</u>. Let $a \in P$, x, $y \in F(S_a)$. Then

$$[S_a,x][S_a,y]e_a = \sum_{\sigma\in W_a} [S_a,xy_\sigma]e_a.$$

<u>Proof</u>. Let $\{a_j : 1 \le j \le n\} \subseteq P$, α, α_i, K_i be as in Lemma 5.3. Denote $S = S_a^1 \dot{\cup} \dots \dot{\cup} S_a^{V_a} \dot{\cup} \dot{\bigcup}_{j=1}^{n} S_{a_j}$, and let $\pi_i : S_a \times S_a \to S_a$ be the coordinate projections, $i = 1, 2$. Using the additivity of F, together with 5.2 and 5.3 we have

$$\begin{aligned}
[S_a,x][S_a,y]e_a &= [S_a \times S_a, \pi_1^0(x) \cdot \pi_2^0(y)]e_a \\
&= [S,\alpha^0(\pi_1^0(x) \cdot \pi_2^0(y))]e_a \\
&= \sum_{i=1}^{V_a} [S_a^i, K_i^0\alpha^0(\pi_1^0(x) \cdot \pi_2^0(y))]e_a \\
&= \sum_{i=1}^{V_a} [S_a^i, \alpha_i^0(\pi_i^0(x) \cdot \pi_2^0(y))]e_a \\
&= \sum_{i=1}^{V_a} [S_a^i, (\pi_1\alpha_i)^0(x) \cdot (\pi_2\alpha_i)^0(y)]e_a
\end{aligned}$$

$$= \sum_{i=1}^{V_a} [S_a^i, x \cdot (\sigma_i)^0(y)] e_a$$

$$= \sum_{\sigma \in W_a} [S_a, xy_\sigma] e_a. \qquad \square$$

<u>Corollary 5.5</u>. Suppose $a \in P$, $x \in F(S_a)$, $y \in F_N(S_a)$. Then $[S_a,x][s_a,y]e_a = V_a[S_a,xy]e_a$.

The following lemma will be crucial in computing the prime ideals of $A_F(G)$, when $F \in AM^G$ is normal.

Lemma 5.6. Let $a \in P$, $x \in F(S_a)$, and $y \in F_N(S_a)$. Then, $[S_a,x][S_a,y] = V_a[S_a,xy] + \sum_{j=1}^{n} [S_{a_j},x_j]$, where $a_j < a$, $x_j \in F(S_{a_j})$, all j.

<u>Proof</u>. Let $\{a_j : 1 \leq j \leq n\} \subseteq P$, α, α_i, K_i be as in 5.3, and let $S = S_a^1 \dot{\cup} \ldots \dot{\cup} S_a^{V_a} \dot{\cup} \dot{\bigcup}_{j=1}^{n} S_{a_j}$. Let $\pi_i : S_a \times S_a \to S_a$ be the coordinate projections. By Lemma 5.3, $\pi_1 \alpha K_j : S_a^j = S_a \to S_a$ is the identity map, and $\pi_2 \alpha K_j : S_a^j = S_a \to S_a$ is a G-automorphism, all j. Therefore, $x = (\pi_1 \alpha K_j)^0(x)$, and $y = (\pi_2 \alpha K_j)^0(y)$, all j, since $y \in F_N(S_a)$. Thus,

$$[S_a,x][S_a,y] = [S_a \times S_a, \pi_1^0(x) \cdot \pi_2^0(y)]$$

$$= [S, (\pi_1 \alpha)^0(x) \cdot (\pi_2 \alpha)^0(y)]$$

$$= \sum_{i=1}^{V_a} [S_a^i, (\pi_1\alpha K_i)^0(x)\cdot(\pi_2\alpha K_i)^0(y)]$$

$$+ \sum_{j=1}^{n} [S_{a_j}, x_j], \quad \text{some} \quad x_j \in F(S_{a_j})$$

$$= \sum_{i=1}^{V_a} [S_a, xy] + \sum_{j=1}^{n} [S_{a_j}, x_j]$$

$$= V_a[S_a, xy] + \sum_{j=1}^{n} [S_{a_j}, x_j]. \qquad \square$$

For any monoid H, Let $\mathbb{Q}H$ denote the rational group algebra. For $a \in P$, define $\psi_a : \mathbb{Q}F(S_a) \to \mathbb{Q}A_F(G) \cdot e_a$ by $\psi_a(x) = V_a^{-1}[S_a, x]e_a$, all $x \in F(S_a)$, then extend linearly to all of $\mathbb{Q}F(S_a)$.

Lemma 5.7. For any $a \in P$, ψ_a is a surjective $\mathbb{Q}$-space homomorphism.

Proof. Everything is clear except surjectivity. It is sufficient to show that for any $b \in P$, $x \in F(S_b)$, $[S_b, x]e_a \in \operatorname{im}\psi_a$. We proceed by induction on b. First note that if $a \not\leq b$, then $[S_b, x]e_a = 0 \in \operatorname{im}\psi_a$, by 5.2, and if $a = b$, then $[S_a, x]e_a = \psi_a(V_a x)$. In particular, this covers the case $b = 1$. Assume that $b > 1$, and that whenever $c < b$ and $y \in F(S_c)$, then $[S_c, y]e_a \in \operatorname{im}\psi_a$. We may also assume $a < b$. Applying 5.1 and 2.5(a) we have

$$[S_b,x]e_a = [S_b,x]e_ae_a$$

$$= V_a^{-1}[S_b,x][S_a,1]e_a$$

$$= V_a^{-1}V_{b,a,b}[S_b,x]e_a + V_a^{-1}\sum_{j=1}^{r}[S_{b_j},x_j]e_a,$$

where $b_j < b$, and $x_j \in F(S_{b_j})$, $1 \leq j \leq r$. Since $a < b$, $V_{b,a,b} = 0$ by 2.2(c). By induction, each $[S_{b_j},x_j]e_a \in \mathrm{im}\psi_a$, so that

$$[S_b,x]e_a = V_a^{-1}\sum_{j=1}^{r}[S_{b_j},x_j]e_a \in \mathrm{im}\psi_a. \qquad \square$$

If $S \in \hat{G}$ and $\sigma \in W_S$, then clearly $\sigma \cdot (xy) = (\sigma\cdot x)(\sigma\cdot y)$, all x, $y \in F(S)$. It follows that W_S acts as a group of ring automorphisms on $\mathbb{Q}F(S)$. We let $\mathbb{Q}F(S)^{W_S}$ denote the fixed ring under this action, that is $\mathbb{Q}F(S)^{W_S} = \{x \in \mathbb{Q}F(S):\sigma \cdot x = x,\ \text{all}\ \sigma \in W_S\}$. Then there is a $\mathbb{Q}$-space epimorphism $\rho:\mathbb{Q}F(S) \to \mathbb{Q}F(S)^{W_S}$ given by $\rho(x) = |W_S|^{-1}\sum_{\sigma\in W_S}\sigma \cdot x$. Note that the restriction of ρ to $\mathbb{Q}F(S)^{W_S}$ is the identity; moreover, $\rho(x\rho(y)) = \rho(x)\rho(y)$, all x, $y \in \mathbb{Q}F(S)$. If $a \in P$ and $S = S_a$, we denote $\rho_a = \rho$. Thus $\rho_a(x) = V_a^{-1}\sum_{\sigma\in W_a}\sigma \cdot x$, all $x \in \mathbb{Q}F(S_a)$.

<u>Proposition 5.8</u>. Let $a \in P$. Then $\psi_a \rho_a = \psi_a$. Moreover, the map $\chi_a : \mathbb{Q}F(S_a)^{W_a} \to \mathbb{Q}A_F(G)e_a$, given by $\chi_a(x) = \psi_a(x)$, all $x \in \mathbb{Q}F(S_a)^{W_a}$, is a surjective $\mathbb{Q}$-algebra homomorphism.

<u>Proof</u>. If $\sigma \in W_a$, then by Corollary 3.4 $[S_a,x] = [S_a,x_\sigma]$, all $x \in F(S_a)$, hence $\psi_a(x) = \psi_a(x_\sigma)$, all $x \in F(S_a)$. But then, $\psi_a\rho_a(x) = V_a^{-1} \sum_{\sigma \in W_a} \psi_a(x_\sigma) = V_a^{-1} \sum_{\sigma \in W_a} \psi_a(x) = \psi_a(x)$. The first result follows, since $F(S_a)$ spans $\mathbb{Q}F(S_a)$. Furthermore the surjectivity of ψ_a, together with $\psi_a\rho_a = \psi_a$, imply that χ_a is surjective. To see that χ_a is an algebra homorphism, let x, $y \in F(S_a)$. Then

$$\begin{aligned}
\chi_a(\rho(x)\rho(y)) &= \chi_a(\rho(x \cdot \rho(y))) \\
&= \psi_a(x \cdot \rho(y)) \\
&= V_a^{-1} \sum_{\sigma \in W_a} \psi_a(xy_\sigma) \\
&= V_a^{-2} \sum_{\sigma \in W_a} [S_a,xy_\sigma]e_a \\
&= V_a^{-2}[S_a,x][S_a,y]e_a \quad \text{(by 5.4)} \\
&= (V_a^{-1}[S_a,x]e_a)(V_a^{-1}[S_a,x]e_a) \\
&= \psi_a(x) \cdot \psi_a(y) = \chi_a(\rho(x)) \cdot \chi_a(\rho(y)).
\end{aligned}$$

Since the elements $\{\rho(x): x \in F(S_a)\}$ span $\mathbb{Q}F(S_a)^{W_a}$, χ_a is a $\mathbb{Q}$-algebra homomorphism, as asserted. $\square$

As we shall presently show, each χ_a is an isomorphism.

Lemma 5.9. Let $S \in G$ and $x, y \in F(S)$. Then $x \sim_S y$ if and only if $\rho(x) = \rho(y)$.

Proof $\Rightarrow$). Suppose there is some $\alpha \in W_S$ with $\alpha^0(x) = y$. Then

$$\rho(y) = |W_S|^{-1} \sum_{\sigma \in W_S} \sigma^0(y) = |W_S|^{-1} \sum_{\sigma \in W_S} \sigma^0 \alpha^0(x)$$

$$= |W_S|^{-1} \sum_{\sigma \in W_S} (\alpha\sigma)^0(x) = |W_S|^{-1} \sum_{\sigma \in W_S} \sigma^0(x) = \rho(x).$$

$\Leftarrow$) Suppose $\rho(x) = \rho(y)$. Since $F(S)$ is a $\mathbb{Q}$-basis of $\mathbb{Q}F(S)$, the identity $\sum_{\sigma \in W_S} \sigma^0(x) = \sum_{\sigma \in W_S} \sigma^0(y)$ implies that $\sigma^0(x) = \tau^0(y)$ for some $\sigma, \tau \in W_S$. If $\alpha = \sigma\tau^{-1} \in W_S$, then $\alpha^0(x) = y$. Thus $x \sim_S y$. $\square$

Lemma 5.10. Let $x_1, \ldots, x_n \in F(S_a)$, with $x_i \not\sim_a x_j$ if $i \neq j$. Then $\{[S_a, x_i]e_a : 1 \le i \le n\}$ is a linearly independent set in $\mathbb{Q}A_F(G)e_a$.

Proof. For any i, Lemma 5.1 implies that $[S_a, x_i]e_a = \sum_{b \le a} \lambda_{b,a}[S_a, x_i][S_b, 1] = \lambda_{a,a}[S_a, x_i][S_a, 1]$

$+ \sum_{b<a} \lambda_{b,a}[S_a,x_i][S_b,1] = [S_a,x_i] + \sum_{j=1}^{n} c_j[S_{a_j},y_j]$, some $a_j < a \in P$, $c_j \in \mathbb{Q}$, $y_j \in F(S_{a_j})$. Therefore, if there is a dependence relation $\sum_{i=1}^{n} d_i[S_a,x_i]e_a = 0$ (where $d_i \in \mathbb{Z}$ without loss of generality), then Corollary 3.4 together with the above yields a dependence relation $\sum_{i=1}^{n} d_i[S_a,x_i] = 0$. By Theorem 3.9, and the assumption on the x_i, it follows that $d_i = 0$, all i. □

<u>Theorem 5.11</u>. For any $a \in P$, the map $\chi_a : \mathbb{Q}F(S_a)^{W_a} \to \mathbb{Q}A_F(G)e_a$ is a $\mathbb{Q}$-algebra isomorphism.

<u>Proof</u>. All that remains is injectivity. If R_a is a set of representatives for $\sim_a$ in $F(S_a)$, then by Lemma 5.9, $\{\rho_a(x) : x \in R_a\}$ spans $\mathbb{Q}F(S_a)^{W_a}$ as a $\mathbb{Q}$-space. By Lemma 5.10, the set $\{\chi_a\rho_a(x) : x \in R_a\} = \{V_a^{-1}[S_a,x]e_a : x \in R_a\}$ is linearly independent over $\mathbb{Q}$. The result follows. □

<u>Theorem 5.12</u>. Let G be a finite group, and let $F : \hat{G} \to AM$ be an additive contravariant functor. Then the injections $\chi_a : \mathbb{Q}F(S_a)^{W_a} \to \mathbb{Q}A_F(G)e_a$ induce a $\mathbb{Q}$-algebra isomorphism

$$\chi = (\chi_a) : \prod_{a \in P} \mathbb{Q}F(S_a)^{W_a} \to \mathbb{Q}A_F(G).$$

In particular, if every transitive G-set is normal over F, then

$$\mathbb{Q}A_F(G) \cong \prod_{a\in P} \mathbb{Q}F(S_a).$$

We remark that the only denominators used in the proof that χ is an isomorphism were divisors of powers of $|G|$. Thus this theorem is valid upon replacing $\mathbb{Q}$ by any field K, where $\mathrm{char}(K) \nmid |G|$.

Several ring theoretic properties of $\mathbb{Q}A_F(G)$ now become transparent. We single out the following.

Corollary 5.13. Suppose $F:\hat{G} \to AB$ is an additive contravariant functor satisfying

i) for all $S \in \hat{G}$, $F(S)$ is torsion, and

ii) every transitive G-set is normal over F. Then $\mathbb{Q}A_F(G)$ is a von Neumann regular ring.

Proof. If $S \in \hat{G}$, then $F(S)$ is a torsion abelian group, hence it is locally finite. By a theorem of Villamayor (1958), the group algebra $\mathbb{Q}F(S)$ is von Neumann regular. Since the product of regular rings is again a regular ring, the result follows from the second part of Theorem 5.12. □

Corollary 5.14. If $F:\hat{G} \to AB$ is any contravariant additive functor, then $J(\mathbb{Q}A_F(G)) = 0$.

Proof. By a result of Montgomery (1976), if R is any ring acted upon by a finite group W of ring automorphisms, and if $|W|^{-1} \in R$, then $J(R^W) = J(R) \cap R^W$. Applying this to $R = \mathbb{Q}F(S_a)$ and $W = W_a$, it follows from Passman (1971,

p. 73) that $J(\mathbb{Q}F(S_a)^{W_a}) = 0$. Since the radical respects products of rings, the result is a direct consequence of Theorem 5.12. □

The Structure of $\mathbb{Q}A_F(G/H)$

Theorem 5.12 effectively computes $\mathbb{Q}A_F(G)$. We shall now indicate a construction which will permit the computation of $\mathbb{Q}A_F(G/H)$, for any subgroup $H \leq G$.

Definition 5.15. If $H \leq G$ and S is an H-set, then the fibered product of G with S, denoted $Gx^H S$, is the G-set of (equivalence classes of) pairs (g,s), where $g \in G$, $s \in S$, with the identification $(g,s) = (gh^{-1},hs)$, all $h \in H$. The G-action on $Gx^H S$ arises from multiplication in the first component.

The notation $Gx^H S$ is not standard; this is usually written as $Gx_H S$. However, we have already used the later to denote to the pullback of G/H-sets. Thus, to avoid ambiguity, we will be non-standard.

Given two H-sets S and T, and an H-map $\phi:S \to T$, the map $1x^H\phi:Gx^H S \to Gx^H T$ given by $(1x^H\phi)(g,s) = (g,\phi(s))$ is a well-defined G-map.

Lemma 5.16. The correspondences $S \to Gx^H S$, and $\phi \to 1x^H\phi$ define a covariant, sum preserving functor from $\hat{H}$ to $\hat{G}$.

<u>Proof</u>. To say that $Gx^H(*)$ is sum preserving is to say that, given any H-sets S and T, there is a natural isomorphism of G-sets $(Gx^H S) \dot{\cup} (Gx^H T) \cong Gx^H(S \dot{\cup} T)$. This is clear. $\square$

It follows that if $F \in AM^G$, then $F \circ Gx^H(*) \in AM^H$. For notation, let $F_H = F \circ Gx^H(*)$. Thus, for any H-set S, $F_H(S) = F(Gx^H S)$, and for any H-map $\phi: S \to T$, $F_H(\phi) = (1x^H\phi)^0 : F_H(T) \to F_H(S)$. The result we are after is to show that for any subgroup $H \leq G$, there is an isomorphism between $A_F(G/H)$ and $A_{F_H}(H/H)$. We must first introduce some notation.

Let $H \leq G$, and let S be a G-set. Suppose there is a G-map $\alpha: S \to G/H$. Denote by $S_\alpha = \{x \in S : \alpha(x) = 1H\}$. Plainly, S_α is an H-set. Denote by μ_α the G-map $Gx^H S_\alpha \to S$ given by $\mu_\alpha(g,s) = g \cdot s$, all $(g,s) \in Gx^H S_\alpha$. It follows easily that μ_α is a G-isomorphism. Indeed, we are just formalizing the well-known fact that the categories of H-sets and G-sets over G/H are equivalent. Define a function $\Lambda_H = \Lambda : A_F(G/H) \to A_{F_H}(H/H)$ by $\Lambda([S,\alpha,x]) = [S_\alpha, \mu_\alpha^0(x)]_H$, where we use the notation $[*,*]_H$ to denote elements of $A_{F_H}(H/H)$. Since $x \in F(S)$, $\mu_\alpha^0(x) \in F(Gx^H S_\alpha) = F_H(S_\alpha)$, so our definition makes sense. We are ready to attack the main result of this section.

Theorem 5.17. For any functor $F \in AM^G$, and subgroup $H \leq G$, the function $\Lambda_H : A_F(G/H) \to A_{F_H}(H/H)$ is a ring isomorphism.

Proof. We shall show that Λ is a well defined bijection, and leave the straightforward verification that Λ preserves sums and products to the reader. Let $[S,\alpha,x]$, $[T,\beta,y]$ $\in A_F(G/H)$.

i) Λ is well defined. If $(S,\alpha,x) \cong (T,\beta,y)$, then choose a G-isomorphism $\phi : S \to T$ with $\alpha = \beta\phi$ and $\phi^0(y) = x$. We must show $(S_\alpha, \mu_\alpha^0(x)) \cong (T_\beta, \mu_\beta^0(y))$ in (H,F_H). Note that if $s \in S_\alpha$, then $\beta\phi(s) = \alpha(s) = 1H$, so $\phi(s) \in T_\beta$. Similarly, if $t \in T_\beta$, then $\phi^{-1}(t) \in S_\alpha$. Thus, $\psi = \phi|_{S_\alpha}$ is an H-isomorphism $S_\alpha \to T_\beta$. We claim $\mu_\beta(1x^H\psi) = \phi\mu_\alpha$. Indeed, if $(g,s) \in Gx^HS_\alpha$, then $\mu_\beta(1x^H\psi)(g,s) = \mu_\beta(g,\phi(s))$ $= g\phi(s) = \phi(gs) = \phi\mu_\alpha(g,s)$. Thus $(1x^H\psi)^0\mu_\beta^0(y) = \mu_\alpha^0\phi^0(y)$ $= \mu_\alpha^0(x)$. It follows that $\psi : (S_\alpha, \mu_\alpha^0(x)) \to (T_\beta, \mu_\beta^0(y))$ is an isomorphism.

ii) Λ is injective. Suppose that $\Lambda([S,\alpha,x])$ $= \Lambda([T,\beta,y])$, that is $[S_\alpha, \mu_\alpha^0(x)]_H = [T_\beta, \mu_\beta^0(y)]_H$. By Corollary 3.4, there is an H-isomorphism $\psi : S_\alpha \to T_\beta$ with $(1x^H\psi)^0\mu_\beta^0(y) = \mu_\alpha^0(x)$. Let $\phi = \mu_\beta \circ (1x^H\psi) \circ \mu_\alpha^{-1} : S \to T$. Then ϕ is a G-isomorphism, with $\phi^0(y)$ $= (\mu_\alpha^{-1})^0(1x^H\psi)^0(\mu_\beta)^0(y) = (\mu_\alpha^{-1})^0(\mu_\alpha)^0(x) = x$. To see that $\alpha = \beta\phi$, let $s \in S$, and choose $g \in G$ with $\alpha(s) = gH$.

Then $\beta\phi(s) = \beta\mu_\beta(1x^H\psi)\mu_\alpha^{-1}(s) = \beta\mu_\beta(1x^H\psi)(g,g^{-1}s)$ $= \beta\mu_\beta(g,\psi(g^{-1}s)) = \beta(g\psi(g^{-1}s)) = g\beta(\psi(g^{-1}s)) = gH = \alpha(s)$. Thus $\phi:(S,\alpha,x) \to (T,\beta,y)$ is an isomorphism.

iii) Λ is surjective. Let $[T,y]_H \in A_{F_H}(H/H)$. Denote $S = Gx^HT$, and define $\alpha:S \to G/H$ by $\alpha(g,t) = gH$. Then α is a well defined G-map with $S_\alpha = \{(h,t) \in Gx^HT: h \in H, t \in T\}$. Since $(h,t) = (1,ht)$, the map $\psi:S_\alpha \to T$ given by $\psi(h,t) = ht$ is an H-isomorphism. Moreover, $\mu_\alpha = 1x^H\psi:Gx^HS_\alpha \to S$. Thus $[T,y]_H = [S_\alpha,(1x^H\psi)^0(y)]$ $= [S_\alpha,\mu_\alpha^0(y)] = \Lambda([S,\alpha,y])$. $\square$

We can combine this result with Theorem 5.12 to determine the structure of $\mathbb{Q}A_F(G/H)$. If $K \leq H \leq G$, then there is an embedding $\theta:\mathrm{Aut}_H(H/K) \to \mathrm{Aut}_G(G/K)$ given by $\theta(\phi)(gK) = g\phi(1K)$, all $\phi \in \mathrm{Aut}_H(H/K)$, $gK \in G/K$. Denote by $W_K^H = \{\theta(\phi):\phi \in \mathrm{Aut}_H(H/K)\} = \mathrm{im}\theta$. Upon identifying $\mathrm{Aut}_H(H/K)$ with $N_H(K)/K$ and $\mathrm{Aut}_G(G/K)$ with $N_G(K)/K$, it is easy to see that θ corresponds to the inclusion of $N_H(K)/K$ into $N_G(K)/K$. As before, W_K^H will act on the group $F(G/K)$, and thus also act on the group algebra $\mathbb{Q}F(G/K)$.

Theorem 5.18. Let $F:\hat{G} \to AM$ be an additive contravariant functor. Let $H \leq G$. Denote by $P(H)$ a set of representatives of conjugacy classes of subgroups of H. Then there is a $\mathbb{Q}$-algebra isomorphism: $\mathbb{Q}A_F(G/H) \cong \prod_{K\in P(H)} \mathbb{Q}F(G/K)^{W_K^H}$.

Proof. For $K \in P(H)$, set $W_K = \{1x^H\phi : \phi \in \mathrm{Aut}_H(H/K)\} \subseteq \mathrm{Aut}_G(Gx^H H/K)$. By 5.12 and 5.17, $\mathbb{Q}A_F(G/H) \cong \mathbb{Q}A_{F_H}(H/H) \cong \prod_{K\in P(H)} \mathbb{Q}F_H(H/K)^{W_K} = \prod_{K\in P(H)} \mathbb{Q}F(Gx^H H/K)^{W_K}$. However, for any $K \in P(H)$, $Gx^H H/K \cong G/K$ (via $(g,hK) \to ghK$). Furthermore, this isomorphism carries the automorphism $1x^H\phi$ of $Gx^H H/K$ to the automorphism $\theta(\phi)$ of G/K; hence, it carries W_K onto W_K^H. It follows that $\mathbb{Q}F(Gx^H H/K)^{W_K} \cong \mathbb{Q}F(G/K)^{W_K^H}$, for each K. □

Corollary 5.19. If F is any additive contravariant functor from $\hat{G}$ to AB, and $S \in \hat{G}$, then $J(\mathbb{Q}A_F(S)) = 0$.

Proof. Expressing S as a disjoint union of transitive G-sets, the result follows directly from 3.6, 5.14, and 5.17. □

CHAPTER 6

PRIME IDEALS IN THE F-BURNSIDE RING

Throughout this chapter we fix a finite group G, and a functor $F \in AM^G$ such that every transitive G-set is normal over F. In this setting, most of the structural results for $A(G)$ can be extended in some fashion to $A_F(G)$. The object of this chapter is to illustrate this principle.

An Embedding Theorem for $A_F(G)$

For $a \in P$, we let $\chi_a:\mathbb{Q}F(S_a) \to \mathbb{Q}A_F(G)e_a$ be the isomorphism of Chapter 5. Thus, $\chi_a(x) = V_a^{-1}[S_a,x]e_a$, all $x \in F(S_a)$. By Theorem 5.12, the product map $\chi = (\chi_a)$: $\prod_{a\in P} \mathbb{Q}F(S_a) \to \mathbb{Q}A_F(G)$ is an isomorphism. We let $\Gamma:\mathbb{Q}A_F(G) \to \prod_{a\in P} \mathbb{Q}F(S_a)$ be the inverse of χ. For $b \in P$, we have the projection homomorphism $r_b: \prod_{a\in P} \mathbb{Q}F(S_a) \to \mathbb{Q}F(S_b)$. We denote by Γ_b the composition $\Gamma_b = r_b\Gamma:\mathbb{Q}A_F(G) \to \mathbb{Q}F(S_b)$. Evidently, each Γ_b is a surjective $\mathbb{Q}$-algebra homorphism.

Lemma 6.1. For any $a \in P$ we have

(a) $\Gamma([S_a,x]e_a) = V_a x$, all $x \in F(S_a)$,

(b) $\Gamma_a\chi_a$ is the identity on $\mathbb{Q}F(S_a)$,

(c) $\chi_a\Gamma_a(x) = xe_a$, all $x \in \mathbb{Q}A_F(G)$.

Proof. (a) If $x \in F(S_a)$, then $x = \Gamma\chi(x)$ $= \Gamma(V_a^{-1}[S_a,x]e_a)$.

(b) Since $\chi|_{\mathbb{Q}F(S_a)} = \chi_a$, it follows that for any $x \in \mathbb{Q}F(S_a)$, $\Gamma_a\chi_a(x) = r_a\Gamma\chi(x) = r_a(x) = x$.

(c) Let $x \in \mathbb{Q}A_F(G)$. Then $x = x \cdot 1 = \sum_{b\in P} xe_b$, where $xe_b \in \mathbb{Q}A_F(G)e_b$. By Lemma 5.7, there are elements $y_b \in \mathbb{Q}F(S_b)$ such that $\chi_b(y_b) = xe_b$, all $b \in P$. Then $\chi_a\Gamma_a(x) = \chi_a\Gamma_a\left(\sum_{b\in P} xe_b\right) = \chi_a\Gamma_a\left(\sum_{b\in P} \chi_b(y_b)\right) = \chi_a r_a\Gamma\chi\left(\sum_{b\in P} y_b\right)$ $= \chi_a r_a\left(\sum_{b\in P} y_b\right) = \chi_a(y_a) = xe_a$. □

Lemma 6.2. Let $g = |G|$, and suppose $0 \neq n \in \mathbb{Z}$ satisfies $g^2|n$. Then for any $a \in P$ and $x \in F(S_a)$, $nx \in \Gamma_a(A_F(G))$.

Proof. Write $n = g^2m$, some $m \in \mathbb{Z}$. By 6.1(c), $\chi_a(nx)$ $= nV_a^{-1}[S_a,x]e_a = nV_a^{-1}[S_a,x]e_a^2 = \chi_a\Gamma_a(nV_a^{-1}[S_a,x]e_a)$. By the injectivity of χ_a, $nx = \Gamma_a(nV_a^{-1}[S_a,x]e_a)$. By 2.2(a), (b), it follows that $gV_a^{-1} \in \mathbb{Z}$, and $ge_a \in A(G) \subseteq A_F(G)$; hence, $nV_a^{-1}[S_a,x]e_a = m(gV_a^{-1})[S_a,x](ge_a) \in A_F(G)$. Thus, $nx \in \Gamma_a(A_F(G))$. □

Lemma 6.3. $\Gamma(A_F(G)) \subseteq \prod_{a\in P} \mathbb{Z}F(S_a)$.

Proof. Let $b \in P$ and $x \in F(S_b)$. By 3.11(c) it suffices to show that $\Gamma_a([S_b,x]) \in \mathbb{Z}F(S_a)$, all $a \in P$. By 6.1(c) and 2.5(a), $\chi_a\Gamma_a([S_b,x]) = [S_b,x]e_a = V_a^{-1}[S_b,x][S_a,1]e_a$ $= V_a^{-1}[S_b x S_a, \pi_b^0(x)]e_a$. By 2.2(c), $S_b \times S_a$ is a union of

$V_{b,a}$ copies of S_a, together with various other S_c, where $c < a$. Thus, using the additivity of F together with 5.2, it follows that

$$\chi_a\Gamma_a([S_b,x]) = V_a^{-1}[S_b \times S_a,\pi_b^0(x)]e_a$$

$$= \sum_{i=1}^{V_{b,a}} V_a^{-1}[S_a,x_i]e_a, \quad \text{some} \quad x_i \in F(S_a), \quad 1 \leq i \leq V_{b,a}$$

$$= \sum_{i=1}^{V_{b,a}} \chi_a(x_i) = \chi_a\left(\sum_{i=1}^{V_{b,a}} x_i\right).$$

Since χ_a is injective, $\Gamma_a([S_b,x]) = \sum_{i=1}^{V_{b,a}} x_i \in \mathbb{Z}F(S_a)$. □

Combining these lemmas, we obtain the following theorem.

<u>Theorem 6.4</u>. $\prod_{a\in P} (|G|^2\mathbb{Z})F(S_a) \subseteq \Gamma(A_F(G)) \subseteq \prod_{a\in P} \mathbb{Z}F(S_a)$.

<u>Corollary 6.5</u>. The group $\prod_{a\in P} \mathbb{Z}F(S_a)/\Gamma(A_F(G))$ is $|G|^2$-torsion.

<u>Prime Ideals</u>

We wish to compute almost all of the prime ideals of $A_F(G)$. Note that when $F = I$, the proof of Lemma 6.3 shows that $\Gamma_a([S_b,1]) = V_{b,a}$, all a, $b \in P$. Especially, the set of maps $\{\Gamma_a : a \in P\}$ is the same set used by Dress (1969) to describe the prime ideals of $A(G) \cong A_I(G)$. Following his notation, for any $a \in P$, and prime $0 < p \in \mathbb{Z}$, we let $q(a,p) = \{x \in A(G) : \Gamma_a(x) \equiv 0 \pmod p\}$, and for $p = 0$, $q(a,0) = \ker\Gamma_a$. The following description of the prime ideals of $A(G)$ is sufficient for our purposes.

<u>Proposition 6.6</u>. Let q be a prime ideal of $A(G)$. Then there is a unique minimal element $a \in P$ (w.r.t. $\leq$) such that if $p = \operatorname{char}(A(G)/q)$,

(a) $q = q(a,p)$,

(b) for any $b < a \in P$, $[S_b,1] \in q(a,p)$,

(c) $[S_a,1] \notin q(a,p)$.

<u>Proof</u>. Dress (1969, p. 215). □

When the prime ideal q of $A(G)$ is written in the form $q(a,p)$, where $a \in P$ is the element given in the Proposition, we will say that q is in <u>standard form</u>. Note that this form is unique: if $q(a,p) = q(b,p')$ are both in standard form, then $a = b$ and $p = p'$. We now extend this result to $A_F(G)$.

<u>Proposition 6.7</u>. Let Q be a prime ideal of $A_F(G)$, such that $Q \cap \mathbb{Z} = p\mathbb{Z}$, where $p \nmid |G|$ (possibly $p = 0$). Then there is a unique minimal element $a \in P$ (w.r.t. $\leq$) such that

(a) for any $b < a \in P$, and any $x \in F(S_b)$, $[S_b, x] \in Q$,

(b) for any $x \in F(S_a)$, if also $x^{-1} \in F(S_a)$, then $[S_a, x] \notin Q$.

<u>Proof</u>. Since $Q \cap A(G)$ is a prime ideal of $A(G)$ lying over $p\mathbf{Z}$, we may apply 6.6, and write $Q \cap A(G) = q(a,p)$, in standard form.

(a) Induce on $b < a \in P$. If $a = 1$, the result is clear, so we may assume $a > 1$. By 6.6(b), $[S_b, 1] \in Q$, all $b < a$. If $b = 1$, then by 5.6, $[S_1, x][S_1, 1] = V_1[S_1, x]$. Since $[S_1, 1] \in Q$ and $p \nmid V_1$, it follows that $[S_1, x] \in Q$. For the induction step, assume $1 < b < a$, and that for all $c < b$, $y \in F(S_c)$, we have $[S_c, y] \in Q$. Then, by 5.6, $[S_b, x][S_b, 1] = V_b[S_b, x] + \sum_{j=1}^{n} [S_{b_j}, y_j]$, where $b_j < b$, $y_j \in F(S_{b_j})$ all j. But $[S_b, 1] \in Q$, and all $[S_{b_j}, y_j] \in Q$ by the induction hypothesis. Therefore $V_b[S_b, x] \in Q$, and since $p \nmid V_b$, $[S_b, x] \in Q$.

(b) If in fact $[S_a, x] \in Q$, then by Lemma 5.6, $[S_a, x][S_a, x^{-1}] = V_a[S_a, 1] + \sum_{j=1}^{n} [S_{a_j}, x_j]$, where $a_j < a$, $x_j \in F(S_{a_j})$. By part (a), all $[S_{a_j}, x_j] \in Q$. But this implies that $V_a[S_a, 1] \in Q \cap A(G) = q(a,p)$, which contradicts $p \nmid V_a$ and $[S_a, 1] \notin Q$.

If $b \in P$ also satisfies (a) and (b), then $[S_a, 1][S_b, 1] = \sum_{c \leq a, b} V_{a,b,c}[S_c, 1] \notin Q$. Thus, there is some $c \leq a$, b such that $[S_c, 1] \notin Q$. By (a), $a = b = c$. ☐

For a prime ideal Q of $A_F(G)$ such that $Q \cap \mathbb{Z} = p\mathbb{Z}$, where $p \nmid |G|$, let $a \in P$ be the element given in the Proposition. Define $V(a,Q) = \{x \in \mathbb{Z}F(S_a) : |G|^n x \in \Gamma_a(Q),\ \text{some } n \geq 0\}$.

Lemma 6.8. In the setting above, $V(a,Q)$ is a prime ideal of $\mathbb{Z}F(S_a)$ lying over $p\mathbb{Z}$.

Proof. Set $g = |G|$. To see that $V(a,Q)$ is an ideal, let $x \in V(a,Q)$, $y \in \mathbb{Z}F(S_a)$. Say $g^n x = \Gamma_a(z)$, some $z \in Q$, $n \geq 0$. By Lemma 6.2, $g^2 y = \Gamma_a(w)$, some $w \in A_F(G)$. Then $g^{n+2}(xy) = g^n x g^2 y = \Gamma_a(z) \cdot \Gamma_a(w) = \Gamma_a(zw)$, so $xy \in V(a,Q)$. Since $V(a,Q)$ is additively closed, it is an ideal of $\mathbb{Z}F(S_a)$. To see that $V(a,Q)$ is prime, let x, $y \in \mathbb{Z}F(S_a)$ with $g^n(xy) = \Gamma_a(z)$, some $z \in Q$, $n \geq 0$. Since $ge_a \in A_F(G)$, $\chi_a(g^{n+4}xy) = g^4\chi_a(g^n xy) = g^4\chi_a\Gamma_a(z) = g^3 z(ge_a) \in Q$. As in 6.2, it follows that $\chi_a(g^2x)$, $\chi_a(g^{n+2}y) \in A_F(G)$, thus, one of $\chi_a(g^2x)$, $\chi_a(g^{n+2}y) \in Q$. Applying Γ_a, and 6.1(b), either $g^2x \in \Gamma_a(Q)$ or $g^{n+2}y \in \Gamma_a(Q)$, that is, $x \in V(a,Q)$ or $y \in V(a,Q)$. Thus $V(a,Q)$ is a prime ideal.

To see that $V(a,Q) \cap \mathbb{Z} = p\mathbb{Z}$, first let $x = t \cdot 1_{F(S_a)} \in V(a,Q) \cap \mathbb{Z}$, with $t \in \mathbb{Z}$. Say $g^n x = \Gamma_a(z)$, $n \geq 0$, $z \in Q$. Then $\chi_a(g^{n+1}x) = g^{n+1}t\chi_a(1_{F(S_a)}) = g^{n+1}te_a = g^{n+1}tV_a^{-1}[S_a,1] + g^n t \sum_{b<a} g\lambda_{b,a}[S_b,1]$. On the other hand,

$\chi_a(g^{n+1}x) = \chi_a\Gamma_a(gz) = (ge_a)z \in Q$, so it follows from the choice of $a \in P$ and Proposition 6.6, that $g^n t(gV_a^{-1})[S_a,1] \in Q$. Again by 6.6, $[S_a,1] \notin Q$, so $g^n t(gV_a^{-1}) \in Q \cap \mathbb{Z} = p\mathbb{Z}$. Since $p \nmid |G|$, we must have $p|t$, establishing the inclusion $V(a,Q) \cap \mathbb{Z} \subseteq p\mathbb{Z}$. Conversely, since $Q \cap \mathbb{Z} = p\mathbb{Z}$, $p[S_a,1] \in Q$. Then, using $\Gamma_a([S_a,1]) = V_a 1_{F(S_a)}$, we have $g(p \cdot 1_{F(S_a)}) = (gV_a^{-1})pV_a 1_{F(S_a)} = \Gamma_a((gV_a^{-1})p[S_a,1]) \in \Gamma_a(Q)$, so $p \cdot 1_{F(S_a)} \in V(a,Q) \cap \mathbb{Z}$. The result follows. □

For an $a \in P$ and prime ideal L of $\mathbb{Z}F(S_a)$, define $R(a,L) = \{x \in A_F(G):\Gamma_a(x) \in L\} = \Gamma_a^{-1}(L) \cap A_F(G)$. Plainly, $R(a,L)$ is a prime ideal of $A_F(G)$.

<u>Lemma 6.9</u>. Let Q be a prime ideal of $A_F(G)$ such that $Q \cap \mathbb{Z} = p\mathbb{Z}$, where $p \nmid |G|$. Let $a \in P$ be the element given in Proposition 6.7. Then $Q = R(a,V(a,Q))$.

<u>Proof</u>. We must show that $Q = \{x \in A_F(G):\Gamma_a(x) \in V(a,Q)\}$.

$\subseteq$) Let $x \in Q$. Then $\Gamma_a(x) \in \Gamma_a(Q)$, so $\Gamma_a(x) \in V(a,Q)$.

$\supseteq$) Let $x \in A_F(G)$ be such that $\Gamma_a(x) \in V(a,Q)$. If $g = |G|$, then $g^n\Gamma_a(x) = \Gamma_a(y)$, some $y \in Q$, $n \geq 0$. It follows that $(ge_a)g^n x = \chi_a\Gamma_a(g^{n+1}x) = \chi_a\Gamma_a(gy) = (ge_a)y \in Q$. However, $g \in Q$ (since $p \nmid g$), and therefore by 6.6 and the choice of $a \in P$, $ge_a \notin Q$. Thus $x \in Q$. □

Lemma 6.10. Let $a \in P$, and let L be a prime ideal of $\mathbb{Z}F(S_a)$ with $L \cap \mathbb{Z} = p\mathbb{Z}$. Then $R(a,L) \cap A(G) = q(a,p)$.

Proof. $\subseteq$) If $x \in R(a,L) \cap A(G)$, then $\Gamma_a(x) \in L \cap \mathbf{Z} = p\mathbf{Z}$, so $x \in q(a,p)$.

$\supseteq$) If $x \in q(a,p)$, then $\Gamma_a(x) \in p\mathbf{Z} \subseteq L$, so $x \in R(a,L) \cap A(G)$. □

Lemma 6.11. Let $a \in P$. Suppose L_1, L_2 are prime ideals of $\mathbb{Z}F(S_a)$, where $L_i \cap \mathbf{Z} = p_i\mathbf{Z}$, $p_i \nmid |G|$, $i = 1, 2$. If $R(a,L_1) = R(a,L_2)$, then $L_1 = L_2$.

Proof. Set $g = |G|$, and let $x \in L_1$. Then $g^2x = \Gamma_a(y)$, some $y \in A_F(G)$, by Lemma 6.2. Since $y \in R(a,L_1) = R(a,L_2)$, we have $g^2x = \Gamma_a(y) \in L_2$. Since $p_2 \nmid g$, we conclude that $x \in L_2$, establishing $L_1 \subseteq L_2$. By a symmetrical argument, $L_2 \subseteq L_1$. □

Theorem 6.12. Let $F:\hat{G} \to AM$ be a contravariant additive functor such that every transitive G-set is normal over F. Let $q(a,p)$ be a prime ideal of $A(G)$ in standard form, with $p \nmid |G|$. Then there is a bijective correspondence between the set of prime ideals of $A_F(G)$ lying over $q(a,p)$ and the set of prime ideals of $\mathbf{Z}F(S_a)$ lying over $p\mathbf{Z}$.

Proof. If L is a prime ideal of $\mathbb{Z}F(S_a)$ lying over $p\mathbf{Z}$, then by 6.10, $R(a,L)$ is a prime ideal of $A_F(G)$ lying over $q(a,p)$. The correspondence $L \to R(a,L)$ is injective by 6.11, and surjective by 6.8 and 6.9. □

The Extension $A_F(G)/A(G)$

We shall finish this chapter by describing those normal functors F for which the ring extension $A_F(G)/A(G)$ is integral. Indeed, this will occur precisely when each of the groups $F(S)$, $S \in G$, is torsion.

For any integer $n > 0$, we let S^n denote the product of n copies of S, and $\pi_{n,i}: S^n \to S$ will denote projection onto the ith component.

Lemma 6.13. Let F be a normal functor, $S \in G$ and $x \in F(S)$. Then $[S,x]^n = [S^n, \pi^0_{n,1}(x^n)]$, for all $0 < n \in \mathbf{Z}$.

Proof. Induction on n. The formula being clear for $n = 1$, assume $n > 1$, and that the result holds for lesser n. Let $t: S^n \to S^n$ be the G-automorphism which interchanges the first two components, and is the identity on every other component. Clearly, $\pi_{n,2} = \pi_{n,1}t$, so by normality of F, $\pi^0_{n,2}(y) = t^0\pi^0_{n,1}(y) = \pi^0_{n,1}(y)$, all $y \in S$. Thus,

$$[S,x]^n = [S,x][S^{n-1}, \pi^0_{n-1,1}(x^{n-1})]$$

$$= [S^n, \pi^0_{n,1}(x) \cdot \pi^0_{n,2}(x^{n-1})]$$

$$= [S^n, \pi^0_{n,1}(x) \cdot \pi^0_{n,1}(x^{n-1})] = [S^n, \pi^0_{n,1}(x^n)]. \qquad \square$$

Theorem 6.14. Let $F \in AM^G$ be a normal functor. Then the extension $A_F(G)/A(G)$ is integral if and only if for every G-set S, $F(S)$ is a torsion group.

Proof $\Rightarrow$). Assume $A_F(G)/A(G)$ is integral. Since $A(G)/\mathbf{Z}$ is already integral, so is $A_F(G)/\mathbf{Z}$. By way of contradiction suppose that for some G-set S, $F(S)$ is not torsion. By additivity of F, this implies that for some $a \in P$, $F(S_a)$ is not torsion. Pick $x \in F(S_a)$ of infinite order. Since $x^j \neq x^k$ if $j \neq k$, normality of F implies that $x^j \not\sim_a x^k$. By integrality, choose $\Phi(X) = X^n + \sum_{k=0}^{n-1} c_k X^k \in \mathbf{Z}[X]$ with $\Phi([S_a,x]) = 0$, that is $[S_a,x]^n + \sum_{k=1}^{n-1} c_k [S_a,x]^k + c_0 = 0$. Multiplying both sides by $V_a e_a$ and applying 5.5 and 2.5(a), this yields

$$V_a^n[S_a,x^n]e_a + \sum_{k=1}^{n-1} c_k V_a^k [S_a,x^k]e_a + c_0[S_a,1]e_a = 0,$$

which is a contradiction to 5.10. $\square$

$\Leftarrow$) By 3.11(c), it suffices to show that if $a \in P$ and $x \in F(S_a)$, then $[S_a,x]$ is integral over $A(G)$. If (say) $x^n = 1$, then by 6.13, $[S_a,x]^n = [S_a^n, \pi^0_{n,1}(x^n)] = [S_a^n,1] \in A(G)$. Thus $[S_a,x]$ satisfies the monic polynomial $X^n - [S_a^n,1] \in A(G)[X]$. $\square$

Finally, we wish to make a statement about the Boolean algebra of idempotents of $A_F(G)$.

Theorem 6.15. Let $F:\hat{G} \to AB$ be a normal, additive, contravariant functor. Then $A(G)$ and $A_F(G)$ contain exactly the same idempotents.

Proof. By a theorem of Kaplansky (see Passman 1971), given any group H, the only idempotents in the group algebra $\mathbb{Z}H$ are 0, 1. Thus if $e \in A_F(G)$ is idempotent, then for any $b \in P$, $\Gamma_b(e) \in \{0,1\}$. Therefore $\Gamma(e) \in \prod_{a \in P} \mathbb{Z} \cdot 1_{F(S_a)}$. It follows from the definition of χ that, $e = \chi\Gamma(e) \in \mathbb{Q}A(G)$. Since also $e \in A_F(G)$, we must have $e \in A(G)$. The other inclusion is trivial. $\square$

CHAPTER 7

THE BRAUER RING OF A FIELD

In this chapter we begin the study of the tensor product of separable algebras over a field. Our guiding question is this: is there a natural ring into which we may embed the Brauer group as a subgroup of its unit group? Of course, one should expect this ring to yield information about separable algebras which the Brauer group does not, and one should hope to be able to recover the Brauer group from purely ring theoretic properties. Although the material presented here may seem unrelated to what has come before, the necessary tie up will come next chapter. We begin our discussion with a generalization of a well known result on the tensor product of two subfields of a finite Galois extension.

Tensor Products of Separable Algebras

Let R be a commutative ring, with $0, 1$ its only idempotents (R is connected). Let S be a Galois extension of R, and let S_1, S_2 be separable, G-strong subalgebras of S, where G is the Galois group of S/R (see Chase, Harrison and Rosenberg (1965) for definitions). Let $H_i \leq G$ be the Galois group of S/S_i, $i = 1, 2$. Choose

$\sigma_1, \ldots, \sigma_m \in G$ to obtain a double coset decomposition $G = \bigcup_{i=1}^{m} H_1\sigma_i H_2$. Define a map $\phi: S_1 \otimes_R S_2 \to S_1 S_2^{\sigma_1} \dotplus \ldots \dotplus S_1 S_2^{\sigma_m}$ by $\phi(u \otimes v) = (u\sigma_1(v), \ldots, u\sigma_m(v))$, where $S_1 S_2^{\sigma_i}$ denotes the compositum of S_1 and $\sigma_i(S_2)$ in S. Plainly, ϕ is a well defined R-algebra homomorphism.

<u>Proposition 7.1</u>. ϕ is an injective R-algebra homorphism.

<u>Proof</u>. Suppose $\sum_i u_i \otimes v_i \in \ker\phi$, so that $\sum_i u_i\sigma_j(v_i) = 0$ for all $1 \leq j \leq m$. Let $\tau \in G$; find $\alpha \in H_1$, $\beta \in H_2$ so that $\tau = \alpha\sigma_j\beta$ for some j. Then $\sum_i u_i\tau(v_i) = \alpha(\sum_i u_i\sigma_j(v_i)) = 0$, showing

(1) $$\sum_i u_i\tau(v_i) = 0 \quad \text{for all} \quad \tau \in G.$$

If E denotes the S-algebra of all functions from G to S under pointwise operations, then the map $h: S \otimes_R S \to E$ given by $h(u \otimes v)(\sigma) = u \cdot \sigma(v)$ is an S-algebra isomorphism, by Chase et al. (1965, p. 4). By (1), $\sum_i u_i \otimes v_i \in \ker h = 0$. □

Under rather non-restrictive conditions, ϕ will also be surjective.

<u>Proposition 7.2</u>. Let $g = |G|$. Suppose that $g = g \cdot 1_R$ is a unit in R. Then ϕ is an isomorphism.

<u>Proof</u>. Since S/R is Galois, there are elements $x_1, \ldots, x_n;\ y_1, \ldots, y_n$ of S such that $\sum_{i=1}^{n} x_i\sigma(y_i) = \delta_{1,\sigma}$, all $\sigma \in G$. Set $x_i' = \sum_{\rho\in H_1} \rho(x_i)$ and $y_{ij}' = \sum_{\gamma\in H_2} \gamma\sigma_j^{-1}(y_i)$. By Galois theory, $x_i' \in S_1$, $y_{ij}' \in S_2$, $1 \leq i \leq n$, $1 \leq j \leq m$. Set $g_k = |H_1 \cap \sigma_k H_2 \sigma_k^{-1}|$, $1 \leq k \leq m$. We claim that

$$\sum_{i=1}^{n} x_i'\sigma_k(y_{ij}') = g_k\delta_{j,k}, \quad 1 \leq j,\ k \leq m. \tag{2}$$

Indeed,

$$\begin{aligned}
\textstyle\sum_i x_i'\sigma_k(y_{ij}') &= \textstyle\sum_i \sum_{\rho\in H_1} \sum_{\rho\in H_2} \rho(x_i)\sigma_k\gamma\sigma_j^{-1}(y_i) \\
&= \textstyle\sum_{\rho\in H_1} \sum_{\rho\in H_2} \rho(\sum_i x_i\rho^{-1}\sigma_k\gamma\sigma_j^{-1}(y_i)) \\
&= \textstyle\sum_{\rho\in H_1} \sum_{\rho\in H_2} \delta_{1,\rho^{-1}\sigma_k\gamma\sigma_j^{-1}}
\end{aligned}$$

by the condition on the x_i and y_i. If $j \neq k$, then σ_j and σ_k are distinct double coset representatives, so that $\rho^{-1}\sigma_k\gamma\sigma_j^{-1} \neq 1$, all ρ and γ, and (2) holds in this case. If $j = k$, then $\rho^{-1}\sigma_k\gamma\sigma_j^{-1} = 1$ iff $\rho = \sigma_k\gamma\sigma_k^{-1} \in H_1 \cap \sigma_k H_2 \sigma_k^{-1}$. Since ρ uniquely determines γ, (2) holds in all cases.

Since g_k divides g, our hypothesis implies that g_k is a unit in R, all k. Define $e_k = g_k^{-1}\sum_i x_i' \otimes y_{ik}'$

$\in S_1 \otimes_R S_2$. From (2) it follows that $\phi(e_k) = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the kth place. The surjectivity of ϕ follows easily. □

We remark that implicit in the proof of 7.2 is the construction of the indecomposable idempotents of $S_1 \otimes_R S_2$, namely, the e_k. When R and S are both fields, there is an easier proof.

<u>Proposition 7.3</u>. If R and S are fields, then ϕ is an isomorphism.

<u>Proof</u>. Since $\operatorname{im}\phi \subseteq S_1 S_2^{\sigma_1} \dot{+} \cdots \dot{+} S_1 S_2^{\sigma_m}$, we may prove equality by counting dimensions. Since ϕ is injective, $\dim(\operatorname{im}\phi) = (\dim S_1) \cdot (\dim S_2)$. Moreover, by Rotman (1978, p. 17), $\dim(S_1 S_2^{\sigma_1} \dot{+} \cdots \dot{+} S_1 S_2^{\sigma_m}) = \sum_{i=1}^{m} [S_1 S_2^{\sigma_i} : R]$

$= \sum_{i=1}^{m} [G:H_1 \cap {}^{\sigma_i}H_2] = [G:H_1][G:H_2] = (\dim S_1) \cdot (\dim S_2)$. □

<u>The Brauer Ring</u>

Let E/F be a (not necessarily finite) Galois extension of fields. Let SEP(E,F) be the category of separable F-algebras A, with the center of A (denoted Z(A)) isomorphic with a finite product of finite dimensional subfields of E. If the extension E/F is understood, we abbreviate SEP(E,F) as SEP. Plainly, SEP is closed under the formation of algebra products. It is also closed under tensor products. Indeed, let A, $B \in$ SEP with $Z(A) \cong K_1 \dotplus \dots \dotplus K_m$ and $Z(B) = L_1 \dotplus \dots \dotplus L_n$ with each K_i, L_j a finite separable extension of F. Since any pair K_i, L_j can be embedded in a finite Galois extension of F contained in E, it follows that $Z(A \otimes_F B) \cong Z(A) \otimes_F Z(B) \cong \prod_{i,j} K_i \otimes_F L_j$, which in turn is isomorphic with a finite product of finite dimensional subfields of E by Proposition 7.3. It follows that we may form the associated Grothendieck-ring of this category, as in Bass (1968, pp. 344-47). Thus, denote $S(E,F) = K_0\mathrm{SEP}(E,F)$. We denote the image of an object $A \in \mathrm{SEP}(E,F)$ in $S(E,F)$ by $[A]$. The following proposition collects some basic facts.

Proposition 7.4. (a) For elements $[A]$, $[B]$ in $S(E,F)$, $[A] + [B] = [A + B]$ and $[A][B] = [A \otimes_F B]$. Also, $1_{S(E,F)} = [F]$.

(b) Every element of $S(E,F)$ can be written in the form $[A] - [B]$ for some A, $B \in$ SEP.

(c) If A, $B \in$ SEP, then $[A] = [B]$ if and only if $A \cong B$ as F-algebras.

Proof. (a), (b) and the if part of (c) are direct consequences of the definitions. For the only if part of (c), suppose $[A] = [B]$. Then there is an algebra $C \in$ SEP with $A \dotplus C \cong B \dotplus C$ as F-algebras. Since separable F-algebras are finite dimensional and semisimple, the uniqueness statement of Wedderburn's theorem implies that $A \cong B$ as F-algebras. $\square$

Note that if A is a finite dimensional, simple F-algebra, with $Z(A)$ isomorphic to a (finite separable) subfield of E, then in particular, A is a central simple $Z(A)$-algebra, and $Z(A)$ is a separable F-algebra. Since central simple algebras are separable, transitivity of separability implies that A is a separable F-algebra. It follows that $A \in$ SEP. Thus any product of matrix algebras, $M_{n_1}(D_1) \dot{+} \cdots \dot{+} M_{n_r}(D_r)$, with each D_i a division algebra, and $Z(D_i)$ isomorphic to a finite dimensional subfield of E, is in SEP. Conversely, by Wedderburn's theorem, any algebra $B \in$ SEP has this form uniquely up to F-isomorphism. This discussion, together with 7.4, establishes the following proposition.

<u>Proposition 7.5</u>. As an abelian group, $S(E,F)$ is free on the set $\{[A]:A \in \text{SEP},\ A \text{ is simple}\}$.

<u>Proposition 7.6</u>. There is a group endomorphism β of $S(E,F)$ such that if $A \cong M_n(D)$ as F-algebras, where $D \in$ SEP is a division algebra, then $\beta([A]) = [D]$. The image of β is the subgroup of $S(E,F)$ that is freely generated by $\{[D]:D \in \text{SEP is a division algebra}\}$. Moreover, for all u, $v \in S(E,F)$, we have $\beta(\beta(u)) = \beta(u)$ and $\beta(uv) = \beta(\beta(u) \cdot \beta(v))$.

Proof. If $A \in SEP$ is simple, then $A \cong M_n(D)$, where D is a division algebra with $Z(A) \cong Z(D)$. Thus, $D \in SEP$. Moreover, if $B \cong M_m(D') \in SEP$ with $A \sim B$, then by Wedderburns theoerm, $D \cong D'$. It follows from 7.5 that the correspondence $[A] \to [D]$ gives a well defined group endomorphism β of $S(E,F)$ such that $\beta([M_n(D)]) = [D]$. The statement regarding the image of β is clear. Since $\beta(\beta([M_n(D)])) = \beta([D]) = [D] = \beta([M_n(D)])$, it follows from 7.5 that $\beta^2(u) = \beta(u)$, all $u \in S(E,F)$. Finally, let $A \cong M_n(D)$, $B \cong M_m(D')$ be in SEP, where D, D' are division algebras. Since $D \otimes_F D'$ is semisimple, we can write $D \otimes_F D' \cong M_{n_1}(D_1) \dotplus \dots \dotplus M_{n_r}(D_r)$. Then $A \otimes_F B \cong (M_n(F) \otimes_F D) \otimes_F (M_m(F) \otimes_F D') \cong M_{nm}(F) \otimes_F (M_{n_1}(D_1) \dotplus \dots \dotplus M_{n_r}(D_r)) \cong M_{nmn_1}(D_1) \dotplus \dots \dotplus M_{nmn_r}(D_r)$. Therefore, $\beta([A][B]) = \beta([A \otimes_F B]) = [D_1] \dotplus \dots \dotplus [D_r] = \beta([D] \cdot [D']) = \beta(\beta([A]) \cdot \beta([B]))$. Again by 7.5, it follows that $\beta(uv) = \beta(\beta(u) \cdot \beta(v))$, for all u, $v \in S(E,F)$. □

Corollary 7.7. $\ker\beta$ is an ideal of $S(E,F)$. As an ideal it is generated by $\{[M_n(F)] - [F] : n \in \mathbf{Z}^+\}$.

Proof. Suppose $u \in \ker\beta$ and $v \in S(E,F)$. Then $\beta(uv) = \beta(\beta(u) \cdot \beta(v)) = \beta(0 \cdot \beta(v)) = 0$, so $uv \in \ker\beta$, and $\ker\beta$ is an ideal. Let I be the ideal of $S(E,F)$ generated by $\{[M_n(F)] - [F] : n \in \mathbf{Z}^+\}$. Plainly, $I \subseteq \ker\beta$. On the other hand, if $A \cong M_n(D) \in SEP$, then $[A] - \beta([A])$ $= [D]([M_n(F)] - [F]) \in I$. Extending linearly, it follows that $[A] - \beta([A]) \in I$ for all $A \in SEP$. Thus, if $[A] - [B] \in \ker\beta$, then $\beta([A]) = \beta([B])$, so that $[A] - [B]$ $= ([A] - \beta([A])) - ([B] - \beta([B])) \in I$. Thus $\ker\beta \subseteq I$. □

The factor ring $S(E,F)/\ker\beta$ is called the <u>Brauer ring</u> of E/F. We denote this ring by $BS(E,F)$. For $A \in SEP$, we denote $\langle A\rangle = [A] + \ker\beta \in BS(E,F)$.

<u>Proposition 7.8</u>. As an abelian group, $BS(E,F)$ is free on the generating set $\{\langle D\rangle : D \in SEP$ is a division algebra$\}$. Moreover, if D, $D' \in SEP$ are division algebras, then $\langle D\rangle = \langle D'\rangle$ if and only if $D \cong D'$ as F-algebras.

<u>Proof</u>. Since $\beta^2 = \beta$, it follows that $S(E,F) = \ker\beta \oplus \operatorname{im}\beta$. Therefore, the canonical isomorphism $BS(E,F) \cong \operatorname{im}\beta$ of abelian groups, together with 7.6, imply the first statement. If $\langle D\rangle = \langle D'\rangle$, then $[D] - [D'] \in \ker\beta \Rightarrow 0 = \beta([D] - [D'])$ $= [D] - [D']$. Thus, $D \cong D'$ as F-algebras by 7.4(c). □

For the field F, let F_S denote its separable algebraic closure. In this case we denote $S(F_S,F) = S(F)$, and $BS(F) = BS(F_S,F)$. Whenever $E \subseteq E'$ is an inclusion of Galois extensions of F, there is a natural inclusion of categories $SEP(E,F) \subseteq SEP(E',F)$, hence also of rings, $S(E,F) \subseteq S(E',F)$, $BS(E,F) \subseteq BS(E',F)$. Since every finite

Galois extension of F is contained in F_S, and F_S is the union (direct limit) of such extensions, we obtain the following.

Proposition 7.9. Let F be any field. Then as rings,

$$S(F) = \bigcup_E S(E,F) = \lim_{\vec{E}} S(E,F),$$

and

$$BS(F) = \bigcup_E BS(E,F) = \lim_{\vec{E}} BS(E,F),$$

where the union and the limit are over the directed set of all finite Galois extensions of F.

Finally, note that the mapping from $Br(F) \to BS(E,F)$, given by $\{A\} \to \langle A \rangle$, is a well defined injection into the group of units of $BS(E,F)$. Indeed, if A and B are central simple F-algebras, with $A \cong M_n(D)$ and $B \cong M_m(D')$ then the equality $\langle A \rangle = \langle B \rangle$ yields $D \cong D'$ as F-algebras, by 7.8. Therefore, $\{A\} = \{B\}$ in $Br(F)$.

Induction and Restriction

We claim that $BS(E,F)$ is the correct ring into which one should embed $Br(F)$. The justification of this assertion is the subject matter of the next chapter. Especially, we shall examine the consequences of the general induction lemma for Mackey-functors. For this, we need a corresponding induction and restriction for the rings $S(E,F)$ and $BS(E,F)$. For any intermediate subfield $F \subseteq K \subseteq E$, we shall let $[A]_K$ denote the image of $A \in SEP(E,K)$ in $S(E,K)$.

Proposition 7.10. Let $F \subseteq K \subseteq L \subseteq E$ be a tower of fields.

(a) There is a group homomorphism $\mathrm{ind} = \mathrm{ind}_{L\to K}$: $S(E,L) \to S(E,K)$ such that $\mathrm{ind}([A]_L) = [A]_K$ for all $A \in SEP(E,L)$.

(b) $\mathrm{ind}_{L\to F} = \mathrm{ind}_{K\to F} \circ \mathrm{ind}_{L\to K}$.

(c) $\mathrm{ind}_{L\to K}$ factors through the projection of S to BS, that is, there is a group homomorphism $\overline{\mathrm{ind}} = \overline{\mathrm{ind}}_{L\to K}$: $BS(E,L) \to BS(E,K)$ such that the following diagram commutes.

$$\begin{tikzcd}
S(E,L) \arrow[r, "\mathrm{ind}"] \arrow[d, "\pi"'] & S(E,K) \arrow[d, "\pi"] \\
BS(E,L) \arrow[r, "\overline{\mathrm{ind}}"'] & BS(E,K)
\end{tikzcd}$$

Proof. (a) This follows from Proposition 7.5, together with the existence of the natural forgetful functor $SEP(E,L) \to SEP(E,K)$.

(b) Clear.

(c) Let $\beta_{L/K}$ denote the endomorphism of $S(L,K)$ given in 7.6. We must show that $\mathrm{ind}_{L\to K}(\ker\beta_{E/L}) \subseteq \ker\beta_{E/K}$.

Suppose $[A]_L - [B]_L \in \ker\beta_{E/L}$. Write $A \cong M_{n_1}(D_1) \dot{+} \dots \dot{+} M_{n_r}(D_r)$ and $B \cong M_{m_1}(D_1') \dot{+} \dots \dot{+} M_{m_s}(D_s')$, where the isomorphisms are as L-algebras. Since $[A]_L - [B]_L \in \ker\beta_{E/L}$, Proposition 7.4(c), together with the uniqueness statement of Wedderburn's theorem, insures $r = s$, and (without loss of generality) $D_i \cong D_i'$ as L-algebras, all i. Then $D_i \cong D_i'$ as K-algebras, all i, so that $[A]_K - [B]_K \in \ker\beta_{E/k}$. □

Restriction will correspond to scalar extension.

<u>Proposition 7.11</u>. Let $F \subseteq K \subseteq L \subseteq E$ be a tower of fields.

(a) There is a ring homomorphism $\mathrm{res} = \mathrm{res}_{K\to L}$: $S(E,K) \to S(E,L)$ such that $\mathrm{res}([A]_K) = [L \otimes_K A]_L$, all $A \in SEP(E,K)$.

(b) $\mathrm{res}_{F\to L} = \mathrm{res}_{K\to L} \circ \mathrm{res}_{F\to K}$.

(v) $\mathrm{res}_{K\to L}$ factors through the projection of S onto BS.

<u>Proof</u>. (a) The existence of res follows the observation that if $A \cong B$ as K-algebras, then $L \otimes_K A \cong L \otimes_K A$ as L-algebras, and 7.5. res is a ring homomorphism because of the distributive property of tensor products over algebra products, and the fact that $L \otimes_K (A \otimes_K B) \cong (L \otimes_K A) \otimes_L (L \otimes_K b)$ as L-algebras.

(b) Clear.

(c) We must show that $\text{res}_{K\to L}(\ker\beta_{E/K}) \subseteq \ker\beta_{E/L}$. Note that if $n \in \mathbb{Z}^+$, then $\text{res}_{K\to L}([M_n(K)]_K - [K]_K) = [M_n(L)]_L - [L]_L$. Therefore, by Corollary 7.7, and the fact that res is a ring homomorphism, the inclusion holds. $\square$

CHAPTER 8

APPLICATIONS OF INDUCTION THEORY

TO ASSOCIATIVE ALGEBRAS

In this chapter we give a construction which allows us to connect the Brauer ring of the previous chapter with the F-Burnside rings we studied earlier. The generality with which this construction goes through gives hope for many more applications than those we include here.

A Category Anti-Equivalence

Fixed throughout this chapter is a finite Galois extension E/F with Galois group $G = \mathrm{Gal}(E/F)$. The category $\hat{G}$ of finite G-sets is then anti-equivalent with the category CSEP(E,F), whose objects are those F-algebras R such that R is F-isomorphic with a finite product of (separable) subfields of E containing F. In other words, CSEP is the full subcategory of SEP consisting of the commutative algebras in SEP. This anti-equivalence is given as follows. For $S \in \hat{G}$, define $R_S = \mathrm{Hom}_G(S,E)$, under pointwise operations. Then $R_S \in$ CSEP. Moreover, if $S \cong G/H$ for some subgroup H of G, then $R_S \cong E^H$ (fixed field of H) under the correspondence $\gamma \to \gamma(1H)$, where $1H$ is the coset containing the identity. For a G-map $\phi: S \to T$, there is an induced F-algebra homomorphism $\phi_*:R_T \to R_S$, given by $\phi_*(\gamma) = \gamma \circ \phi$, all $\gamma \in R_T$. Conversely, if $R \in$ CSEP, define $S_R = \mathrm{Hom}_F(R,E)$, a finite set, which becomes a G-set using the G-action on E. Again we observe that if L is a subfield of E/F, then S_L is isomorphic with the transiive G-set of cosets modulo Gal(E/L). The isomorphism $G/\mathrm{Gal}(E/L) \to S_L$ is given by $\sigma\mathrm{Gal}(E/L) \to \sigma|_L$, any $\sigma \in G$. If $\alpha:R \to R'$ is an F-algebra homomorphism, then the map $\alpha^*:S_{R'} \to S_R$, given by $\alpha^*(f) = f \circ \alpha$ $(f \in S_{R'})$, is a G-map. Note that for any two G-sets S_1, S_2, we have $R_{S_1 \dot{\cup} S_2} = \mathrm{Hom}_G(S_1 \cup S_2,E) \cong \mathrm{Hom}_G(S_1,E) \dot{+} \mathrm{Hom}_G(S_2,E) = R_{S_1} \dot{+} R_{S_2}$. This isomorphism takes an element $\alpha \in R_{S_1 \dot{\cup} S_2}$ to the pair $(\alpha|_{S_1},\alpha|_{S_2})$.

We now show how from an arbitrary covariant, product preserving functor ρ:CSEP $\to$ AM, we may construct an additive contravariant functor $F_\rho:\hat{G} \to$ AM, and thus obtain the Green-functor $A_\rho = A_{F_\rho}$. Namely, define $F_\rho:\hat{G} \to$ AM by $F_\rho(S) = \rho(R_S)$, and for a G-map $\phi:S \to T$, denote (as usual) $\phi^0 = F_\rho(\phi) = \rho(\phi_*):F_\rho(T) \to F_\rho(S)$. Plainly, F_ρ is a contravariant functor from $\hat{G}$ to AM.

<u>Proposition 8.1</u>. Given any covariant, product preserving functor ρ:CSEP(E,F) $\to$ AM, the functor $F_\rho:\hat{G} \to$ AM is additive.

Proof. Let S_1, $S_2 \in \hat{G}$, and let $K_i : S_i \to S_1 \dot{\cup} S_2$ be the inclusions. We must show $K_1^0 \times K_2^0 : F_\rho(S_1 \dot{\cup} S_2) \to F_\rho(S_1) \times F_\rho(S_2)$ is an isomorphism. Let $\theta : R_{S_1 \dot{\cup} S_2} \to R_{S_1} \dot{+} R_{S_2}$ be the canonical isomorphism, and let $\pi_i : R_{S_1} \dot{+} R_{S_2} \to R_{S_i}$ be projection. Since ρ preserves products, the composition $(\rho(\pi_1) \times \rho(\pi_2)) \circ \rho(\theta) : \rho(R_{S_1 \dot{\cup} S_2}) \to \rho(R_{S_1}) \times \rho(R_{S_2})$ is an isomorphism. However, an easy check shows that $K_{i*} = \pi_i \theta$, $i = 1, 2$, so that $(\rho(\pi_1) \times \rho(\pi_2)) \circ \rho(\theta) = \rho(\pi_1 \theta) \times \rho(\pi_2 \theta)$ $= \rho(K_{1*}) \times \rho(K_{2*}) = K_1^0 \times K_2^0$. □

Our applications arise as follows. For any commutative ring R, let $AZ(R)$ denote the category of Azumaya (central separable) R-algebras. When R is a field, $AZ(R)$ coincides with the category of finite dimensional, central simple R-algebras. For an algebra A in $AZ(R)$, let (A) denote its R-algebra isomorphism class, and $\{A\}$ its image in the Brauer group, $Br(R)$. Denote the set of all isomorphim classes in $AZ(R)$ by $AZ_0(R)$. Then $AZ_0(R)$ becomes a commutative monoid under tensor products over R, with identity element (R). If $\phi : R \to S$ is a homomorphism of commutative rings, then the correspondence $(A) \to (S \otimes_R A)$ (where S is considered an R-algebra via ϕ) defines a monoid homomorphism, $AZ_0(R) \to AZ_0(S)$. Thus the correspondence $R \to AZ_0(R)$ defines a covariant functor, which is easily checked to be product preserving (that is,

$AZ_0(R \dot{+} S) \cong AZ_0(R) \times AZ_0(S)$, for any commutative rings R and S). Similarly, the correspondence $R \to Br(R)$ is covariant and product preserving.

By applying Proposition 8.1 to the restrictions of AZ_0 and Br to $CSEP(E,F)$, we may obtain the Green-functors A_{AZ} and A_{Br}. More explicitly, for any G-set S, a typical element of $A_{AZ}(S)$ will be a formal difference $[T_1,\phi_1,(A_1)] - [T_2,\phi_2(A_2)]$, where T_i is a G-set, $\phi_i:T_i \to S$ is a G-map, and $(A_i) \in AZ_0(R_{T_i})$, $i = 1, 2$. A similar description holds for $A_{Br}(S)$. One of the major results of this chapter establishes that for any subgroup $H \leq G$, there are isomorphisms $A_{AZ}(H) \cong S(E,E^H)$ and $A_{Br}(H) \cong BS(E,E^H)$. We first need a few preliminaries on the structure of anti-equivalence of $\hat{G}$ and CSEP.

<u>Proposition 8.2</u>. Let S and T be transitive G-sets, and $\alpha:R_S \to R_T$ an F-algebra isomorphism. Then there is a G-isomorphism $\phi:T \to S$ such that $\phi_* = \alpha$.

<u>Proof</u>. Without loss of generality, we may assume $S = G/H$, and $T = G/J$ for some subgroups $H, J \leq G$. Define $\lambda_S:R_S \to E^H$ by $\lambda_S(\gamma) = \gamma(1H)$ $(\gamma \in R_S)$, and $\lambda_T:R_T \to E^J$ by $\lambda_T(\gamma) = \gamma(1J)$ $(\gamma \in R_T)$. Then λ_S and λ_T are F-algebra isomorphisms. Define $\beta:E^H \to E^J$ by $\beta = \lambda_T\alpha\lambda_S^{-1}$. Thus, if $\gamma \in R_S$, then $\beta\lambda_S(\gamma) = \lambda_T\alpha(\gamma)$, that is,

$\beta\gamma(1H) = \alpha(\gamma)(1J)$. Since E/F is Galois, there exists $\bar{\beta} \in G = \mathrm{Gal}(E/F)$ such that the restriction of $\bar{\beta}$ to E^H is β. Define $\phi: G/J \to G/H$ by $\phi(\sigma J) = \sigma\bar{\beta}H$. Check that this is a well defined G-isomorphism. To see that $\phi_* = \alpha$, let $\gamma \in R_S = \mathrm{Hom}_G(S,E)$ and $t = \sigma J \in T = G/J$. Then $\phi_*(\gamma)(t) = \gamma\phi(\sigma J) = \gamma\sigma\bar{\beta}H = \sigma\bar{\beta}\gamma(1H) = \sigma\beta\gamma(1H) = \sigma\alpha(\gamma)(1J)$ $= \alpha(\gamma)(\sigma J) = \alpha(\gamma)(t)$. □

<u>Proposition 8.3</u>. Let S and T be any G-sets, and suppose $\alpha: R_S \to R_T$ is an F-algebra isomorphism. Then there is a G-isomorphism $\phi: T \to S$ such that $\phi_* = \alpha$.

<u>Proof</u>. Write $S = S_1 \dot{\cup} \ldots \dot{\cup} S_m$ and $T = T_1 \dot{\cup} \ldots \dot{\cup} T_n$ as disjoint unions of transitive G-sets. Since $R_{S_1} \dot{+} \ldots \dot{+} R_{S_m} \cong R_S \cong R_T \cong R_{T_1} \dot{+} \ldots \dot{+} R_{T_n}$, and each R_{S_i}, R_{T_j} is a field, we must have $m = n$. For $1 \leq i \leq n$, let $e_i \in R_S$ and $f_i \in R_T$ be the primitive idempotents corresponding to S_i and T_i, respectively. That is, $e_i(s) = 1$ if $s \in S_i$, and $e_i(s) = 0$ if $s \notin S_i$, and similarly for f_i. Since α is a ring isomorphism, there is a permutation π of $\{1, \ldots, n\}$ such that $\alpha(e_k) = f_{\pi(i)}$. Now, for each i, define $\lambda_i: R_{S_i} \to R_S$ by $\lambda_i(f)(s) = \begin{cases} f(s) & s \in S_i \\ 0 & s \notin S_i \end{cases}$. Then λ_i is a monomorphism with $\lambda_i(1_{R_{S_i}}) = e_i$. Moreover, if

$f \in R_S$, then $\lambda_i(f|_{S_i}) = f \cdot e_i$. Next define $\alpha_i:R_{S_i} \to R_{T_{\pi(i)}}$ by $\alpha_i(f)(t) = \alpha(\lambda_i(f))(t)$, all $t \in T_{\pi(i)}$. It is straightforward to check that each α_i is an F-algebra isomorphism. Thus, by 8.2, there exists $\phi_i:T_{\pi(i)} \to S_i$, G-isomorphisms, with $\phi_{i*} = \alpha_i$. Define $\phi = \phi_1 \dot{\cup} \ldots \dot{\cup} \phi_n:T \to S$. Then ϕ is a G-isomorphism. Moreover, if $f \in R_S$ and $t \in T$ with (say) $t \in T_{\pi(i)}$, then $\phi_*(f)(t) = f\phi(t)$ $= f|_{S_i}(\phi_i(t)) = \phi_{i*}(f|_{S_i})(t) = \alpha_i(f|_{S_i})(t) = \alpha(\lambda_i(f|_{S_i}))(t)$ $= \alpha(fe_i)(t) = \alpha(f)(t) \cdot \alpha(e_i)(t) = \alpha(f)(t)f_{\pi(i)}(t) = \alpha(f)(t)$. Thus, $\phi_* = \alpha$, as needed. □

Proposition 8.4. Suppose α, $\beta:S \to T$ are G-maps, with T a transitive G-set. If $\alpha_* = \beta_*:R_T \to R_S$, then $\alpha = \beta$.

Proof. Without loss of generality $T = G/H$, some subgroup $H \leq G$. Let $s \in S$, and set $\alpha(s) = gH$. By transitivity, there exists $g_1 \in G$ such that $g_1\alpha(s) = \beta(s)$, that is, $\beta(s) = g_1gH$. Since $\alpha_* = \beta_*$, for any $f \in \mathrm{Hom}_G(G/H,E)$ we have $f\alpha = f\beta$. Thus $f(1H) = f(g^{-1}gH) = g^{-1}f(\alpha(s))$ $= g^{-1}f(\beta(s)) = g^{-1}g_1f(\alpha(s)) = g^{-1}g_1gf(1H)$. Since $\mathrm{Hom}_G(G/H,E) \cong E^H$ via the map $f \to f(1H)$, it follows from Galois theory that $g^{-1}g_1g \in H$, hence $g_1g = gh$, some $h \in H$. But then, $\beta(s) = g_1gH = gH = \alpha(s)$. □

Lemma 8.5. Let H, $J \leq G$, and fix a double coset decomposition $G = \dot{\bigcup}_{i=1}^{r} H\sigma_i J$. Then $G/J \cong H/H \cap {}^{\sigma_1}J \,\dot{\cup} \cdots \dot{\cup}\, H/H \cap {}^{\sigma_r}J$ as H-sets.

Proof. For each i, define $\beta_i : H/H \cap {}^{\sigma_i}J \to G/J$ by $\beta_i(h(H \cap {}^{\sigma_i}J)) = h\sigma_i J$. It is straightforward to verify that the map $\beta = \beta_1 \dot{\cup} \cdots \dot{\cup} \beta_r$ is an H-isomorphism. □

Proposition 8.6. Let $H \leq G$. Let S_1, S_2 be any G-sets, and suppose there are G-maps $\alpha_i : S_i \to G/H$, $i = 1, 2$. Define $\phi : R_{S_1} \otimes_{R_{G/H}} R_{S_2} \to R_{S_1 \times_{G/H} S_2}$ by $\phi(f \otimes g)(x,y) = f(x) \cdot g(y)$, all $(x,y) \in S_1 \times_{G/H} S_2$. Then ϕ is an $R_{G/H}$-algebra and $R_{S_1} - R_{S_2}$ bimodule isomorphism.

Proof (Sketch). First suppose S_1 and S_2 are transitive, so that with no loss of generality, $S_1 = G/H_1$ and $S_2 = G/H_2$ for some subgroups H_1, $H_2 \leq G$. Say $\alpha_i(1H_i) = g_i H$, $i = 1, 2$. Then $H_i^{g_i} \subseteq H$, so we may decompose H into $H_1^{g_1} - H_2^{g_2}$ double costs: $H = \dot{\bigcup}_{i=1}^{n} H_1^{g_1}\sigma_i H_2^{g_i}$. Since $R_{G/H_i} \cong E^{H_i} \cong E^{H_i^{g_i}}$ as $R_{G/H}$-algebras, Proposition 7.3 implies $R_{G/H_1} \otimes_{R_{G/H}} R_{G/H_2} \cong E^{H_1^{g_1}} \otimes_{E^H} E^{H_2^{g_2}} \cong \prod_{i=1}^{n} E^{H_1^{g_1}} (E^{H_2^{g_2}})^{\sigma_i}$

$= \prod_{i=1}^{n} E^{H_1^{g_1} \cap H_2^{g_2\sigma_i^{-1}}}$. Explicitly, this map sends $f \otimes g$ to $(\ldots, f(g_1^{-1}H_1) \cdot g(\sigma_i g_2^{-1}H_2), \ldots)$.

Now if an element $\sum_i f_i \otimes g_i \in \ker\Phi$, then $\sum_i f_i(xH_1) \cdot g_i(yH_2) = 0$, whenever $(xH_1, yH_2) \in G/H_1 \times_{G/H} G/H_2$. However, since each $\sigma_i \in H$, it follows that $(g_1^{-1}, \sigma_i g_2^{-1}) \in G/H_1 \times_{G/H} G/H_2$. Thus $\sum_i f_i \otimes g_i$ is in the kernel of the map described in the first paragraph (which was an isomorphism), so $\sum_i f_i \otimes g_i = 0$, and Φ is injective.

Surjectivity of Φ follows from a dimension count. Set $T = \{\sigma \in G : \alpha_1(1H_1) = \alpha_2(\sigma H_2)\}$. If $\sigma \in T$, then $H_1\sigma H_2 \subseteq T$, so we may decompose T into $H_1 - H_2$ double cosets: $T = \dot{\bigcup}_{i=1}^{m} H_1\tau_i H_2$. Set $J_i = H_1 \cap {}^{\tau_i}H_2$, and define $\beta_i : G/J_i \to G/H_1 \times_{G/H} G/H_2$ by $\beta_i(gJ_i) = (gH_1, g\tau_i H_2)$. Five pages of routine calculations show that each β_i is an injective G-map, and that $\beta = \beta_1 \dot{\cup} \ldots \dot{\cup} \beta_m : G/J_1 \dot{\cup} \ldots \dot{\cup} G/J_m \to G/H_1 \times_{G/H} G/H_2$ is a G-isomorphism. Therefore $R_{G/H_1 \times_{G/h} G/H_2} \cong \prod_{i=1}^{m} E^{J_i}$. Another straightforward argument establishes that $m = n$, and that there is a permutation of $\{1, \ldots, n\}$ with $J_{\pi(i)}$ conjugate to $H_1^{g_1} \cap H_2^{g_2\sigma_i^{-1}}$.

In particular, $E^{J_{\pi(i)}} \cong E^{H_1^{g_1} \cap H_2^{g_2 \sigma_i^{-1}}}$, so the dimensions coincide, and Φ is surjective.

In general, write $S_1 = T_1 \dot{\cup} \cdots \dot{\cup} T_r$ and $S_2 = U_1 \dot{\cup} \cdots \dot{\cup} U_t$, as unions of transitive G-sets. Then $R_{S_1} \otimes_{R_{G/H}} R_{S_2} \cong \prod_{i,j} R_{T_i} \otimes_{R_{G/H}} R_{U_j} \cong \prod_{i,j} R_{T_i \times_{G/H} U_j}$

$= R_{\dot{\bigcup}_{i,j} T_i \times_{G/H} U_j} \cong R_{S_1 \times_{G/H} S_2}$. Check that this isomorphism is Φ. □

The Isomorphism Theorem

Let $H \leq G$. For any $S \in \hat{G}$, $A \in AZ(R_S)$ and G-map $\alpha: S \to G/H$, define an $R_{G/H}$-algebra A_α to be A as a ring, with $R_{G/H}$ action induced from $\alpha_*: R_{G/H} \to R_S$. Thus, if $x \in R_{G/H}$ and $a \in A$, then $x \cdot a = \alpha_*(x)a$. Note that $A \cong A_\alpha$ as F-algebras, since α_* is an F-algebra homomorphism.

Proposition 8.7. Let $H \leq G$, and let $[S,\alpha,(A)]$, $[T,\beta,(B)] \in A_{AZ}(H)$. Then $[S,\alpha,(A)] = [T,\beta,(B)]$ if and only if $A_\alpha \cong B_\beta$ as $R_{G/H}$-algebras.

Proof. ⇒) By Corollary 3.4, there is a G-isomorphism $\phi: T \to S$ with $\alpha\phi = \beta$ and $\phi^0((A)) = (B)$. This last condition yields an R_T-algebra isomorphism $\psi: R_T \otimes_{R_S} A \to B$. Define $\gamma: A \to B$ by $\gamma(a) = \psi(1 \otimes a)$, all $a \in A$. Since

$R_S \cong R_T$, α is a ring isomorphism. Furthermore, if $x \in R_{G/H}$, then $\gamma(x \cdot a) = \gamma(\alpha_*(x)a) = \psi(1 \otimes \alpha_*(x)a)$ $= \psi(\phi_*\alpha_*(x) \otimes a) = \psi(\beta_*(x) \otimes a) = \beta_*(x)\psi(1 \otimes a) = \beta_*(x)\gamma(a)$ $= x \cdot \gamma(a)$. Thus γ is an $R_{G/H}$-algebra isomorphism of A_α to B_β.

$\Leftarrow$) Suppose $\gamma: A_\alpha \to B_\beta$ is an $R_{G/H}$ algebra isomorphism. Then $\gamma(Z(A_\alpha)) = Z(B_\beta)$, that is, $\gamma(R_S) = R_T$. By Proposition 8.3, there is a G-isomorphism $\phi: T \to S$ with $\phi_* = \gamma$. We claim that $\alpha\phi = \beta$. By Proposition 8.4, since G/H is transitive, it is enough to show that $\phi_*\alpha_* = \beta_*$: $R_{G/H} \to R_T$. Then, if $x \in R_{G/H}$, we have $\phi_*\alpha_*(x) = \gamma(\alpha_*(x))$ $= \beta_*(x)\gamma(1_A) = \beta_*(x)$. Finally we must show that $\phi^0((A))$ $= (B)$, that is, $R_T \otimes_{R_S} A \cong B$ as R_T-algebras. The map $\psi: R_T \otimes_{R_S} A \to B$ given by $\psi(x \otimes a) = x\gamma(a)$ is such an isomorphism. Thus $\phi: (T,\beta,(B)) \to (S,\alpha,(A))$ is an isomorphism. □

For any subgroup $H \leq G$, the isomorphism $R_{G/H} \cong E^H$ allows us to replace $R_{G/H}$ by E^H, if we consider every $R_{G/H}$ algebra to be an E^H-algebra via this isomorphism. Define $\Psi_H = \Psi: A_{AZ}(H) \to S(E,E^H)$ by $\psi([S,\alpha,(A)]) = [A_\alpha]$. By Proposition 8.7, Ψ is well defined and injective.

<u>Theorem 8.8</u>. For any subgroup $H \leq G$, the map Ψ_H is a ring isomorphism.

Proof. Let $[S,\alpha,(A)]$, $[T,\beta,(B)] \in A_{AZ}(H)$. Since $(A) \dot{+} (B) = (A \dot{+} B)$, and $(A \dot{+} B)_{\alpha \dot{\cup} \beta} \cong A_\alpha \dot{+} B_\beta$ (via the identity), we have $\Psi([S,\alpha,(A)] + [T,\beta,(B)])$
$= \Psi([S \dot{\cup} T, \alpha \dot{\cup} \beta, (A \dot{+} B)]) = [(A \dot{+} B)_{\alpha\dot{\cup}\beta}] = [A_\alpha] + [B_\beta]$
$= \Psi([S,\alpha,(A)]) + \Psi([T,\beta,(B)])$. Now $\Psi([S,\alpha,(A)] \cdot [T,\beta,(B)])$
$= \Psi([S x_{G/H} T, \alpha x_{G/H} \beta, \pi_S^0(A) \cdot \pi_T^0(B)])$, where $\pi_S^0(A) \cdot \pi_T^0(B)$
$= (R_{Sx_{G/H}T} \otimes_{R_S} A) \cdot (R_{Sx_{G/H}T} \otimes_{R_T} B)$
$= (A \otimes_{R_S} R_{Sx_{G/H}T} \otimes_{R_{Sx_{G/H}T}} R_{Sx_{G/H}T} \otimes_{R_T} B)$
$= (A \otimes_{R_S} R_{Sx_{G/H}T} \otimes_{R_T} B) = (A \otimes_{R_S} R_S \otimes_{R_{G/H}} R_T \otimes_{R_T} B)$
$= (A \otimes_{R_{G/H}} B)$, by 8.6. (Note that $A \otimes_{R_{G/H}} B$ is an $R_{Sx_{G/H}T}$ algebra via the composition $R_{Sx_{G/H}T} \to R_S \otimes_{R_{G/H}} R_T$ $\to A \otimes_{R_{G/H}} B$). The identity map: $(A \otimes_{R_{G/H}} B)_{\alpha x_{G/H} \beta}$ $\to A \otimes_{R_{G/H}} B$ is an $R_{G/H}$ algebra isomorphism. Thus,
$\Psi[S,\alpha,(A)] \cdot [T,\beta,(B)]) = [(A \otimes_{R_{G/H}} B)_{\alpha x_{G/H} \beta}]$
$= [A_\alpha \otimes_{R_{G/H}} B_\beta] = [A_\alpha][B_\beta] = \Psi[S,\alpha,(A)]) \cdot \Psi([T,\beta,(B)])$.
To see that Ψ is surjective, let $A \in SEP(E,E^H)$ be simple, with $Z(A) \cong E^J$ for some subgroup $J \leq H$. Let $\alpha: G/J \to G/H$ be projection, that is, $\alpha(gJ) = gH$, all $g \in T$. Then, viewing A as an $R_{G/J}$-algebra via the $R_{G/H}$ isomorphism $R_{G/J} \cong E^J$, we have $A \in AZ(R_{G/J})$. It follows that $\Psi([G/J,\alpha,(A)]) = [A]$. Thus Ψ is surjective by Proposition 7.5 ∎

Following an almost identical proof, we obtain the final result of this section.

Theorem 8.9. Let $H \leq G$. Define a map $A_{Br}(H) \to BS(E,E^H)$ by $[S,\alpha,\{A\}] \to \langle A_\alpha \rangle$. Then this mapping is an isomorphism.

Consequences of the Mackey Induction Lemma

Theorem 8.8 will permit us to apply the induction theory of Mackey-functors to rings $S(E,E^H)$, where $H \leq G$. However, we must first verify that restriction and induction for A_{AZ} and S coincide.

Lemma 8.10. Let $H \leq G$, and set $L = E^H$. Let $\eta: G/H \to G/G$ be the canonical map. Then the following diagrams both commute.

(a)
$$\begin{tikzcd} A_{AZ}(G) \arrow[r, "\eta_*"] \arrow[d, "\Psi_G"'] & A_{AZ}(H) \arrow[d, "\Psi_H"] \\ S(E,F) \arrow[r, "res_{F\to L}"'] & S(E,L) \end{tikzcd}$$

(b)
$$\begin{tikzcd} A_{AZ}(G) \arrow[d, "\Psi_G"'] & A_{AZ}(H) \arrow[l, "\eta^*"'] \arrow[d, "\Psi_H"] \\ S(E,F) & S(E,L) \arrow[l, "ind_{L\to F}"] \end{tikzcd}$$

Proof. (a) It is convenient to identify $R_{G/G}$ with F. If $[S,(A)] \in A_{AZ}(G)$, and if $\pi_S: G/H \times S \to S$ is projection, then by Proposition 8.6, $\pi_S^0(A) = (R_{G/H\times S} \otimes_{R_S} A)$ $= (R_{G/H} \otimes_F A)$. Furthermore, if $\pi_H: G/H \times S \to G/H$ is

projection, then $\pi_{H*}: R_{G/H} \to R_{G/H\times S} \cong R_{G/H} \otimes_F R_S$ is injection, $\pi_{H*}(\gamma) = \gamma \otimes 1$ all $\gamma \in R_{G/H}$. Thus, the identity map $R_{G/H} \otimes_F A \to (R_{G/H} \otimes_F A)_{\pi_H}$ is an $R_{G/H}$ algebra isomorphism. Therefore, $\Psi_H \eta_*([S,(A)]) = \Psi_H([G/H \times S, \pi_H, \pi_S^0(A)])$
$= \Psi_H([G/H \times S, \pi_H, (R_{G/H} \otimes_F A)]) = [(R_{G/H} \otimes_F A)_{\pi_H}]_L$
$= [R_{G/H} \otimes_F A]_L = [L \otimes_F A]_L = \mathrm{res}_{F\to L}([A]_F) = \mathrm{res}_{F\to L}\Psi_G([S,(A)])$.

(b) Let $[S,\alpha,(A)] \in A_Z(H)$. Then $\Psi_G \eta^*([S,\alpha,(A)])$
$= \Psi_G([S,(A)]) = [A]_F = [A_\alpha]_F = \mathrm{ind}_{L\to F}([A_\alpha]_L)$
$= \mathrm{ind}_{L\to F}\Psi_H([S,\alpha,(A)])$, since $A \cong A_\alpha$ as F-algebras. □

We are interested in studying $\ker(\mathrm{res}_{F\to L})$ and $\mathrm{im}(\mathrm{ind}_{L\to F})$; it will be convenient to proceed more generally. Let M be any Mackey-functor $\hat{G} \to AB$, and let S be a G-set. Denote by $K_M(S)$ the kernel of the map $(\eta_S)_*$: $M(G) = M(G/G) \to M(S)$, and by $I_M(S)$ the image of $(\eta_S)^*$: $M(S) \to M(G)$.

<u>Lemma 8.11</u>. (Induction lemma for Mackey-functors.) Let G be a finite group, and $M: \hat{G} \to AB$ a Mackey-functor. Then for any G-set S,

(a) $|G| \cdot (I_M(S) \cap K_M(S)) = 0$,

(b) $|G| \cdot M(G) \subseteq I_M(S) + K_M(S)$.

<u>Proof</u>. See Dress (1971, p. 64). □

In particular, using the commutivity from Lemma 8.10, together with Theorem 3.6 (A_{AZ} is a Green-functor), we obtain

Theorem 8.12. Let E/F be a finite Galois extension with Galois group G. Let $H \leq G$, and set $L = E^H$. Then

(a) $\operatorname{im}(\operatorname{ind}_{L\to F}) \cap \ker(\operatorname{res}_{F\to L}) = 0$,

(b) $|G| \cdot S(E,F) \subseteq \operatorname{im}(\operatorname{ind}_{L\to F}) + \ker(\operatorname{res}_{F\to L})$.

Proof. Take $S = G/H$ in 8.11. We may drop multiplication by $|G|$ in (a) because $S(E,F)$ is free, hence torsion free, by 7.5. The rest is clear. □

Corollary 8.13. Let L/F be a finite separable field extension, and let A and B be separable L-algebras. If $L \otimes_F A \cong L \otimes_F B$ as L-algebras, then $A \cong B$ as F-algebras.

Proof. By a standard characterization of separable algebras over fields, we may write A and B as finite products of finite dimensional, simple L-algebras, where each simple algebra has as center a finite separable field extension of L. Since L/F is finite separable, it follows that there is a finite Galois extension E/F containing the centers of all of these simple algebras. Thus, A, $B \in \operatorname{SEP}(E,L)$. Consider $[A]_F - [B]_F \in S(E,F)$. Plainly, $\operatorname{ind}_{L\to F}([A]_L - [B]_L) = [A]_F - [B]_F$. Also, $\operatorname{res}_{F\to L}([A]_F - [B]_F)$ $= [L \otimes_F A]_L - [L \otimes_F B]_L = 0$, since $L \otimes_F A \cong L \otimes_F B$ as L-algebras. Thus $[A]_F - [B]_F \in \operatorname{im}(\operatorname{ind}_{L\to F}) \cap \ker(\operatorname{res}_{F\to L}) = 0$, so that $[A]_F = [B]_F$. By 7.4(c), $A \cong B$ as F-algebras. □

Of course, this result is not true if A and B do not contain L in their centers. For example, take $F = \mathbb{R}$,

$L = \mathbb{C}$, $A = M_2(\mathbb{R})$ and $B = \mathbb{H}$ (quaternions). Then $A \not\cong B$, but $\mathbb{C} \otimes A \cong M_2(\mathbb{C}) \cong \mathbb{C} \otimes B$, as $\mathbb{C}$-algebras. 8.12(b) yields a much stranger consequence.

Corollary 8.14. Let E/F be a finite Galois extension. Suppose that A is a separable F-algebra whose center is isomorphic with a finite product of subfields of E/F, that is, $A \in SEP(E,F)$. Then there are F-algebras B, $C \in SEP(E,F)$ with $E \otimes_F B \cong E \otimes_F C$ as E-algebras, and there are algebras Y, $Z \in SEP(E,E)$ (that is, finite products of central simple E-algebras) such that $A \dot{+} \ldots \dot{+} A \dot{+} B \dot{+} Y \cong C \dot{+} Z$ as F-algebras, where $[E:F]$ copies of A appear in the left hand product.

Proof. Take $H = \{1\}$ in Theorem 8.12, so that $L = E$. Set $n = [E:F] = |Gal(E/F)|$. Then $n[A]_F \in im(ind_{E \to F}) + ker(res_{F \to E})$, so that $n[A]_F = ind_{E \to F}([Z]_E - [Y]_E) + [C]_F - [B]_F$, where $[C]_F - [B]_F \in ker(res_{F \to E})$, and $[Z]_E - [Y]_E \in S(E,E)$. Thus, $n[A]_F + [Y]_F + [B]_F = [Z]_F + [C]_F$. Using 7.4(c), this translates to the desired result. $\square$

It is worth mentioning that results similar to 8.10 and 8.12 hold upon replacing A_{AZ} by A_{Br} and S by BS. However, these results tell us nothing new, so we will not formulate them precisely.

CHAPTER 9

THE BRAUER RINGS OF $\mathbb{Q}_p$ AND $\mathbb{Q}$

We are ready to combine the results of the preceeding chapters to determine the structure of the ring $\mathbb{Q}BS(E,\mathbb{Q}_p) = \mathbb{Q} \otimes_{\mathbb{Z}} BS(E,\mathbb{Q}_p)$, for a finite Galois extension E of the p-adic rationals $\mathbb{Q}_p$. We shall begin by interpreting normality for the functor F_{Br}.

Normal Algebras

The following definition was first given by Eilenberg and MacLane (1948).

Definition 9.1. Let F be a field, and let L be a finite separable field extension of F. The central simple L-algebra A is normal over F if every F-automorphism of L can be extended to an F-algebra automorphism of A.

As we shall see, if L is a finite separable extension of $\mathbb{Q}_p$, then every central simple L-algebra is normal over $\mathbb{Q}_p$. However, non-normal algebras exist.

For example, let $F = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{2})$, and let A be the generalized quaternion algebra $\left(\frac{-1,-\sqrt{2}}{L}\right)$. Thus, $A = L\cdot 1 \oplus L\cdot i \oplus L\cdot j \oplus L\cdot k$, where $i^2 = -1$, $j^2 = -\sqrt{2}$, and $ij = -ji = k$. Define $\sigma: L \to L$ by $\sigma(\sqrt{2}) = -\sqrt{2}$ and suppose

σ has an extension to an F-algebra automorphism ϕ of A. Set $i_0 = \phi(i)$, $j_0 = \phi(j)$, and $k_0 = \phi(k)$. Since ϕ is F-linear, the set $\{1, i_0, j_0, k_0\}$ is linearly independent over F, from which it follows that $A = L\cdot 1 \oplus L\cdot i_0 \oplus L\cdot j_0 \oplus L\cdot k_0$. However, $i_0^2 = -1$, $j_0^2 = \sqrt{2}$, and $i_0 j_0 = -j_0 i_0 = k_0$, so that $A \cong (\frac{-1,\sqrt{2}}{L})$, that is $(\frac{-1,-\sqrt{2}}{L}) \cong (\frac{-1,\sqrt{2}}{L})$. This isomorphism is impossible since -1 is not the norm of any element of $L(\sqrt{-1})$ to L, that is, $-1 \notin N_{L(\sqrt{-1})/L}(L(\sqrt{-1}))$.

The importance of normal algebras to us is indicated by the following proposition.

Proposition 9.2. Let E/F be a finite Galois extension, with $G = Gal(E/F)$. Let $S \in \hat{G}$ be transitive, and let $\{A\} \in Br(R_S)$. Then $\{A\}$ is a normal element of $F_{Br}(S)$ if and only if A is a normal R_S-algebra over F.

Proof $\Rightarrow$). Let $\alpha \in Aut_F(R_S)$. We must show that α can be extended to A. By Proposition 8.2, we may find $\phi \in Aut_G(S)$ with $\phi_* = \alpha$. Since $\{A\}$ is normal, it follows that $\phi^0(\{A\}) = \{A\}$. But $\phi^0(\{A\}) = \{R_S \otimes_{R_S} A\}$, where R_S is considered as an R_S-algebra via α, that is, $x \cdot y = x\alpha(y)$, for $x, y \in R_S$. By counting dimensions, there is an R_S-algebra isomorphism $\psi: R_S \otimes_{R_S} A \to A$. Define $\gamma: A \to A$ by $\gamma(d) = \psi(1 \otimes d)$, all $d \in A$. If $r \in R_S$, then

$\gamma(rd) = \psi(1 \otimes rd) = \psi(\alpha(r) \otimes d) = \psi(\alpha(r)(1 \otimes d))$ $= \alpha(r)\psi(1 \otimes d) = \alpha(r)\gamma(d)$. Since α is F-linear, γ is an F-algebra isomorphism. Taking $d = 1$ yields $\gamma(r) = \alpha(r)$ all $r \in R_S$, so γ extends α.

<=) Let $\phi \in \mathrm{Aut}_G(S)$. We must show that $\{A\} = \phi_0(\{A\}) = \{R_S \otimes_{R_S} A\}$, where R_S is considered as an R_S-algebra via ϕ_*, as above. Since $\phi_* \in \mathrm{Aut}_F(R_S)$, the normality of A implies the existence of $\alpha \in \mathrm{Aut}_F(A)$ such that $\alpha|_{R_S} = \phi_*$. The map $\psi: R_S \otimes_{R_S} A \to A$, given by $\psi(r \otimes d) = \phi_*^{-1}(r) \cdot d$, is a well defined F-algebra isomorphism. Therefore, $\alpha \circ \psi: R_S \otimes_{R_S} A \to A$ is an R_S-algebra isomorphism, showing $\{R_S \otimes_{R_S} A\} = \{A\}$, as needed. □

The Ring $BS(E, \mathbb{Q}_p)$

For a prime $p \in \mathbb{Z}$, let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ at the p-adic valuation. The next result shows that all finite dimensional simple $\mathbb{Q}_p$-algebras are normal. It is due to Janusz (1978), and the reader may refer to this paper for the proof.

Proposition 9.3. Let $0 \neq p \in \mathbb{Z}$ be a prime. For $i = 1, 2$, let L_i be a finite extension of $\mathbb{Q}_p$, and let A_i be a central simple L_i-algebra. If A_1 and A_2 are isomorphic as rings, then $\mathrm{inv}A_1 = \mathrm{inv}A_2$.

This proposition clearly also holds for $p = \infty$, that is, $\mathbb{Q}_p = \mathbb{R}$. We remark that the notation $\operatorname{inv}A$ for a central simple L-algebra A denotes its Hasse invariant. For a discussion of the properties of this invariant see Pierce (1982). The most important fact for us is that the class of the algebra A in $\operatorname{Br}(L)$ is completely determined by its Hasse invariant.

Corollary 9.4. Let $0 \neq p \in \mathbb{Z}$ be a prime, and let L be a finite extension of $\mathbb{Q}_p$. Then every central simple L-algebra is normal over $\mathbb{Q}_p$.

Proof. Let A be a central simple L-algebra, and let $\alpha \in \operatorname{Aut}_{\mathbb{Q}_p}(L)$. Define an L-algebra B to be A as a ring, with L-algebra structure given by $\ell \cdot b = \alpha(\ell)b$, all $\ell \in L$, $b \in B = A$. Then B is a central simple L-algebra, and $B \cong A$ as rings (in fact as $\mathbb{Q}_p$-algebras). By 9.3, A and B yield the same class in $\operatorname{Br}(L)$. Since $\dim_L A = \dim_L B$, we have $A \cong B$ as L-algebras. Let $\phi: A \to B$ be an L-algebra isomorphism. Using the $\mathbb{Q}_p$-algebra isomorphism $\mathrm{id}: B \to A$, we obtain the $\mathbb{Q}_p$-algebra isomorphism $\gamma = \mathrm{id} \circ \phi: A \to A$. Then, if $\ell \in L$, $\gamma(\ell 1_A) = \phi(\ell 1_A) = \ell \cdot \phi(1_A) = \alpha(\ell) 1_A$, thus γ extends α. □

Corollary 9.5. Let E be a finite Galois extension of $\mathbb{Q}_p$, with $G = \operatorname{Gal}(E/\mathbb{Q}_p)$. Then every transitive G-set is normal over F_{Br}.

Proof. This follows directly from 9.2 and 9.4. □

A mildly surprising result follows from our work. As shown by Eilenberg and MacLane (1948, Corollary 7.3), if E/F is cyclic, then any central simple E-algebra which is normal over F can be obtained by extension of scalars from a central simple F-algebra. Combining this with Corollary 9.4 we obtain the following.

Corollary 9.6. Let $0 \neq p \in \mathbb{Z}$ be a prime, and suppose that $E/\mathbb{Q}_p$ is a finite cyclic Galois extension. Then the canonical homomorphism $BR(\mathbb{Q}_p) \to Br(E)$ is surjective.

Theorem 9.7. Let E be a finite Galois extension of the p-adic field $\mathbb{Q}_p$, and let $G = Gal(E/\mathbb{Q}_p)$. Let $n = |P(G)|$ be the number of conjugacy classes of subgroups of G. Then $\mathbb{Q}BS(E,\mathbb{Q}_p) \cong \Pi_n \mathbb{Q}(\mathbb{Q}/\mathbb{Z})$, where the right hand side is a product of n copies of the group algebra $\mathbb{Q}(\mathbb{Q}/\mathbb{Z})$,

Proof. By Theorem 8.9, $\mathbb{Q}BS(E,\mathbb{Q}_p) \cong \mathbb{Q}A_{Br}(G)$. Since every transitive G-set is normal over F_{Br}, Theorem 5.12 implies that $\mathbb{Q}A_{Br}(G) \cong \prod_{a \in P} \mathbb{Q}Br(R_{S_a})$. However, the Brauer group of a local field is $\mathbb{Q}/\mathbb{Z}$, thus $Br(R_{S_a}) \cong \mathbb{Q}/\mathbb{Z}$ for all $a \in P$. The result follows. □

Passing to direct limits we can state

Proposition 9.8. Let $\bar{\mathbb{Q}}_p$ denote the algebraic closure of $\mathbb{Q}_p$. Then $\mathbb{Q}BS(\bar{\mathbb{Q}}_p,\mathbb{Q}_p)$ is von Neumann regular.

Proof. By Theorem 8.9 and Corollary 5.13, $\mathbb{Q}BS(E,\mathbb{Q}_p)$ is von Neumann regular for each finite Galois extension $E/\mathbb{Q}_p$. The proposition follows from Proposition 7.9 and the fact that the property of being von Neumann regular is preserved under the taking of direct limits. □

The Ring $BS(E,\mathbb{Q})$

Let E be a finite Galois extension of $\mathbb{Q}$. If $p \neq 0$ is an integral prime, then p factors in O_E (the ring of algebraic integers of E) as a product $(P_1 \ldots P_g)^e$. Since each completion E_{P_i} is the compositum of E (embedded in E_{P_i}) and $\mathbb{Q}_p$, the extensions $E_{P_i}/\mathbb{Q}_p$ are all Galois. We introduce the notation E_p to denote the compositum over $\mathbb{Q}_p$ of the Galois extensions $E_{P_1}, \ldots, E_{P_g}$ (E_p is the splitting field over $\mathbb{Q}_p$ of a generating polynomial for the extension $E/\mathbb{Q}$). If $p = \infty$ is the infinite prime, set $E_\infty = \mathbb{R}$ when all of the infinite primes of E are real, otherwise set $E_\infty = \mathbb{C}$. We shall use this notation in attempting the computation of $BS(E,\mathbb{Q})$. We first recall a basic number theory result. Its proof may be found, for example, in Narkiewicz (1974, Proposition 6.1).

Proposition 9.9. Let L be a finite extension of $\mathbb{Q}$ with ring of integers O_L.

(a) Let $0 \neq p \in \mathbb{Z}$ be a prime, and write $pO_L = P_1^{e_1} \cdot \ldots \cdot P_g^{e_g}$ where the P_i are distinct prime of O_L. Then there is a $\mathbb{Q}_p$-algebra isomorphism $L \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong L_{P_1} \dotplus \ldots \dotplus L_{P_g}$.

(b) If the infinite prime of $\mathbb{Q}$ factors into r_1 real and r_2 complex infinite primes in L (so that $r_1 + 2r_2 = [L:\mathbb{Q}]$), then $L \otimes_{\mathbb{Q}} \mathbb{R} \cong \Pi_{r_1} \mathbb{R} \dotplus \Pi_{r_2} \mathbb{C}$.

For a Galois extension e of Q_p we shall use the notation $[\;]_p$, respectively $\langle\;\rangle_p$, to denote elements of $S(E,\mathbb{Q}_p)$, respectively $BS(E,\mathbb{Q}_p)$.

<u>Proposition 9.10</u>. Let $E/\mathbb{Q}$ be a finite Galois extension. For each prime p (possibly infinite) of $\mathbb{Q}$ define a map $\theta_p : S(E,\mathbb{Q}) \to S(E_p,\mathbb{Q}_p)$ by $\theta_p([A]) = [A \otimes_{\mathbb{Q}} \mathbb{Q}_p]_p$. Then

(a) θ_p is a ring homomorphism.

(b) θ_p factors through the projection of S to BS. That is, there is a ring hmomomorphism $\bar{\theta}_p : BS(E,\mathbb{Q}) \to BS(E_p,\mathbb{Q}_p)$ such that the diagram

$$\begin{tikzcd}
S(E,\mathbb{Q}) \arrow[r, "\theta_p"] \arrow[d, "\pi"'] & S(E_p,\mathbb{Q}_p) \arrow[d, "\pi"] \\
BS(E,\mathbb{Q}) \arrow[r, "\bar{\theta}_p"'] & BS(E_p,\mathbb{Q}_p)
\end{tikzcd}$$

commutes.

Proof. (a) If $A \in SEP(E,\mathbb{Q})$ is simple, we may assume without loss of generality that $Z(A) = L$, where $\mathbb{Q} \subseteq L \subseteq E$. Then $A \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong A \otimes_L (L \times_{\mathbb{Q}} \mathbb{Q}_p) \cong A \otimes_L (L_{P_1} \dotplus \cdots \dotplus L_{P_g}) \cong \prod_{i=1}^{g} A \otimes_L L_{P_i}$ as $\mathbb{Q}_p$-algebras, by Proposition 9.9. Since each $A \otimes_L L_{P_i}$ is a central simple L_{P_i}-algebra, and $\mathbb{Q}_p \subseteq L_{P_i} \subseteq E_p$, $A \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is an element of $SEP(E_p,\mathbb{Q}_p)$. It follows from this, together with Proposition 7.5, that θ_p is a well defined group homomorphism. If also $B \in SEP(E,\mathbb{Q}_p)$, then the $\mathbb{Q}_p$-isomorphism $(A \otimes_{\mathbb{Q}} B) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong (A \otimes_{\mathbb{Q}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} (B \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ shows that θ_p is a ring homomorphism. These same arguments work for $p = \infty$.

(b) Let $0 < n \in \mathbf{Z}$. Then $\theta_p([M_n(\mathbb{Q})] - [\mathbb{Q}]) = [M_n(\mathbb{Q}_p)]_p - [\mathbb{Q}_p]_p$. Part (b) then follows from part (a) and Corollary 7.7. $\square$

Patching together the homomorphism of Proposition 9.10 over all primes p, we obtain ring homomorphisms

$$\theta = (\theta_p): S(E,\mathbb{Q}) \to \prod_p S(E_p,\mathbb{Q}_p),$$

and

$$\bar{\theta} = (\bar{\theta}_p): BS(E,\mathbb{Q}) \to \prod_p BS(E_p,\mathbb{Q}_p).$$

The image and kernel of $\bar{\theta}$ are the subject of the remainder of this chapter.

For each prime p (possibly $p = \infty$), let $G_p = \mathrm{Gal}(E_p, \mathbb{Q}_p)$. Then $BS(E_p, \mathbb{Q}_p) \cong A_{Br}(G_p)$. By earlier remarks, the Burnside ring of G_p, $A(G_p)$, can be identified as a subring of $A_{Br}(G_p)$, and thus as a subring of $BS(E_p, \mathbb{Q}_p)$. It is easy to see that $A(G_p)$ correponds to the subring of $BS(E_p, \mathbb{Q}_p)$ consisting of all sums of fields $A(G_p) = \{\sum n_i \langle L_i \rangle_p : n_i \in \mathbf{Z},\ \mathbb{Q}_p \subseteq L_i \subseteq E_p\}$.

Proposition 9.11. Let $\bar{\theta}: BS(E,\mathbb{Q}) \to \prod_p BS(E_p, \mathbb{Q}_p)$ be the ring homomorphism given above. Then $\mathrm{im}\bar{\theta}$ is contained in the restricted direct product of the rings $BS(E_p, \mathbb{Q}_p)$ over the subrings $A(G_p)$.

Proof. The statement of the proposition is equivalent with showing that given any $x \in BS(E,\mathbb{Q})$, one has $\theta_p(x) \in A(G_p)$ for all but finitely many p. Let $A \in SEP(E,\mathbb{Q})$ with A simple, where without loss of generality, $Z(A) = L$, with $\mathbb{Q} \subseteq L \subseteq E$. Now, $A \otimes_L L_P \cong M_n(L_P)$ $(n = \mathrm{Deg}A)$ for all but finitely many primes P of L (see Pierce (1982, Proposition 18.5)), and there are at most finitely many primes of $\mathbb{Q}$ lying under these exceptional primes. If p is not one of them then $A \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{i=1}^{g} A \otimes_L L_{P_i} \cong \prod_{i=1}^{g} M_n(L_{P_i})$, so that

$\bar{\theta}_p(\langle A\rangle) = \sum_{i=1}^{g} \langle M_n(L_{P_i})\rangle_p = \sum_{i=1}^{g} \langle L_{P_i}\rangle_p \in A(G_p)$. Since $BS(E,\mathbb{Q})$ is spanned by the classes $\langle A\rangle$, where A is simple, the result follows from the additivity of $\bar{\theta}$. □

We wish to look at $\ker\bar{\theta}$. For an algebraic number field K, let K_A denote its adele ring. We need a characterization of number fields with isomorphic adele rings.

Proposition 9.12. Let K and L be finite extensions of $\mathbb{Q}$. Denote by V_K the set of non-zero primes of K (including the infinite primes), and similarly for L. Then the following are equivalent.

(1) K_A and L_A are (topologically) isomorphic.

(2) There is a bijection ψ of V_K onto V_L such that given any prime P of K, P and $\psi(P)$ lie over the same prime p of $\mathbb{Q}$, and $K_P \cong L_{\psi(P)}$ as $\mathbb{Q}_p$-algebras.

(3) For every prime p of $\mathbb{Q}$, there is a $\mathbb{Q}_p$-algebra isomorphism $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong L \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Proof. The equivalence (1) ⇔ (2) is given in Komatsu (1978). The equivalence (2) ⇔ (3) follows directly from Proposition 9.9 and the uniqueness statement of Wedderburn's theorem. □

Corollary 9.13. Let $E/\mathbb{Q}$ be a finite Galois extension, and suppose K and L are subfields of E. Then $\langle K\rangle - \langle L\rangle \in \ker\bar{\theta}$ if and only if $K_A \cong L_A$.

Proof. Since K and L are commutative, $\langle K\rangle - \langle L\rangle \in \ker\bar{\theta}$ iff $[K] - [L] \in \ker\theta$ iff $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong L \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for all primes p of $\mathbb{Q}$. Apply the previous proposition. □

At this point the question naturally arises to find nonisomorphic number fields with isomorphic adele rings. An infinite list of such examples was given by Komatsu (1978). We state his result for completeness.

Proposition 9.14. Let m be a square free integer such that $m \neq \pm 1$, ± 2, and $m \equiv 2, 7, 14, 15 \pmod{16}$. Let n be an integer with $n \geq 3$, and set $s = 2^n$. Put $K = \mathbb{Q}(\sqrt[s]{m})$ and $L = \mathbb{Q}(\sqrt{2} \times \sqrt[s]{m})$. Then $K_A \cong L_A$, but K and L are not isomorphic.

We remark that it is an interesting and open problem to classify radical extensions of $\mathbb{Q}$ by the isomorphism type of their adele rings.

If we let I be the ideal of $BS(E,\mathbb{Q})$ generated by the set $\{\langle K\rangle - \langle L\rangle : K_A \cong L_A\}$, then the above shows that $I \subseteq \ker\bar{\theta}$. If $[E:\mathbb{Q}] \leq 6$, or if $E/\mathbb{Q}$ is abelian, then the work of Perlis (1977) establishes that $I = 0$. Hence the equality $I = \ker\bar{\theta}$ would imply the injectivity of $\bar{\theta}$ in these cases. However, it is not known, even when the extensions $E/\mathbb{Q}$ is abelian, whether the inclusion $I \subseteq \ker\bar{\theta}$ is proper or not. Not wishing to conjecture the wrong result, we finish our work here, leaving the foregoing problem unsolved.

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