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INFINITE BRANCHES OF THE PHI-TREE

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Let ϕ denote Euler's totient function. We make the set $\mathscr{J}$ of integers greater than 1 into the vertices of a directed graph, connecting $x \in \mathscr{J}$ with an arrow "pointing down to" $\phi(x)$. For $x \in \mathscr{J}$ with $x \geqslant 3$, the sequence $x, \phi(x), \phi^2(x), \ldots$ eventually reaches 2, hence 2 is the unique minimal element of $\mathscr{J}$. Thus, $\mathscr{J}$ is a tree under this structure, the *Phi-tree*. This idea is due to H. Shapiro [**1**].

For $x, y \in \mathscr{J}$, we say that y is *above* x if there is a directed path from y to x (of course, y is above x if $\phi^k(y) = x$ for some positive integer k). Although the Phi-tree is easily constructed, one may appreciate the difficulty of predicting for given x which y lie above it by trying to find the 77 elements above $x = 40$, for example. Surprisingly, it is easy to determine which x have infinitely many elements of $\mathcal{J}$ above them, given our main theorem.

An even integer of the form $2^e 3^f$ will be called a 2,3-*number*.

THEOREM. *The integer $x \geqslant 2$ has infinitely many elements of the Phi-tree above it if and only if x is a 2,3-number.*

We will give the proof in a sequence of lemmas, beginning with the observations for positive integers e, f that $\phi(2^e 3^f) = 2^e \cdot 3^{f-1}$ and that $\phi(2^e) = 2^{e-1}$. From this the following is clear.

LEMMA 1. *If x is a 2,3-number, then there are infinitely many elements of the Phi-tree above x. Furthermore, every element below x is a 2,3-number.*

Of course we have just established one direction of the theorem.

For an integer y define $\nu(y)$ to be the exponent of 2 in the prime factorization of y. The idea of the rest of the proof is to keep track of the sequence $\nu(y), \nu(\phi(y)), \nu(\phi^2(y)), \ldots$.

LEMMA 2. *Let $y \in \mathcal{J}$ not be a power of 2. Then $\nu(y) \leqslant \nu(\phi(y))$ with equality implying that $y = 2^e p^f$, where e, f are positive integers and p is a prime with $p \equiv 3(4)$. If, in addition, $\nu(y) = \nu(\phi^2(y))$, and y is not a 2,3-number, then $f = 1$.*

Proof. Let p be an odd prime divisor of y and write $y = 2^e p^f m$, where $p \nmid m$ and $e = \nu(y)$. Then

$$\nu(\phi(y)) = \nu(\phi(2^e) p^{f-1}(p-1)\phi(m)) = \nu\left(\phi(2^e) p^{f-1} \cdot 2\frac{p-1}{2} \cdot \phi(m)\right)$$

$$= \nu(\phi(2^e)) + 1 + \nu\left(\frac{p-1}{2}\right) + \nu(\phi(m)).$$

But $\nu(\phi(2^e)) \geqslant e - 1$ with equality provided $e \geqslant 1$. Thus

$$\nu(\phi(y)) \geqslant e - 1 + 1 = \nu(y)$$

as claimed. If $\nu(\phi(y)) = \nu(y)$ then $\nu(\phi(2^e)) = e - 1$, so that $e \geqslant 1$. Also $\nu\left(\frac{p-1}{2}\right) = 0$, so that $p \equiv 3(4)$, and finally $\nu(\phi(m)) = 0$ so that $m = 1$, since m is odd. This completes the proof of the first assertion.

For the second assertion, assume that $y = 2^e p^f$, where p is a prime with $p \equiv 3(4)$. If y is not a 2,3-number, then $p \equiv 3(4)$ implies $p \geqslant 7$ and so $(p-1)/2 \geqslant 3$, whence $2 \left| \phi\left(\frac{p-1}{2}\right)\right.$. But also $(p-1)/2$ is odd, and so in the calculation

$$\phi(y) = 2^e \cdot p^{f-1} \cdot \left(\frac{p-1}{2}\right),$$

the factors 2^e, p^{f-1}, and $(p-1)/2$ are pairwise coprime. Thus

$$\phi^2(y) = \phi(2^e)\phi(p^{f-1})\phi\left(\frac{p-1}{2}\right)$$

and

$$\nu(\phi^2(y)) = e - 1 + \nu(\phi(p^{f-1})) + \nu\left(\phi\left(\frac{p-1}{2}\right)\right)$$

$$\geqslant e - 1 + \nu(\phi(p^{f-1})) + 1 \geqslant \nu(y) + \nu(\phi(p^{f-1})).$$

If now $\nu(\phi^2(y)) = \nu(y)$, then we must have $\nu(\phi(p^{f-1})) = 0$, whence $f = 1$.

The next result, which is fairly well known, we include for the sake of completeness.

LEMMA 3. *For $x \in \mathscr{J}$ the set $\{y|\phi(y)=x\}$ is finite.*

Proof. If $\phi(y)=x$ and if p^e is a prime power divisor of y, then $p^{e-1}(p-1)$ divides x. It follows that $p \leqslant x+1$ and that

$$e \leqslant \log_p(x)+1 \leqslant \log_2(x)+1.$$

A rather crude bound

$$y \leqslant [(x+1)!]^{\log_2(x)+1}$$

follows, and the lemma is thereby proved.

LEMMA 4. *Let $x \in \mathscr{J}$ have infinitely many elements of the Phi-tree above it. Then there is an infinite sequence a_n of integers with $a_1 = x$ and $\phi(a_i) = a_{i-1}$ for all $i \geqslant 2$.*

Proof. Elementary graph theory: Assume for $n \geqslant 1$ that $a_1 = x, a_2, \ldots, a_n$ have been constructed with $\phi(a_i) = a_{i-1}$ for $i \geqslant 2$ and with a_n having infinitely many elements of the Phi-tree above it. By Lemma 3 the set $\{y|\phi(y)=a_n\}$ is finite, and so some such y has infinitely many elements above it. Put $a_{n+1}=y$. Continuing in this way we obtain our sequence.

LEMMA 5. *Let $x \in \mathscr{J}$ have infinitely many elements above it. Then x is a 2,3-number.*

Proof. Given such x, construct a sequence a_n as in Lemma 4. If x is not a 2,3-number, then by Lemma 1, none of the a_n is a 2,3-number. By Lemma 2

$$\nu(a_1) \geqslant \nu(a_2) \geqslant \cdots.$$

Thus there is some $n \geqslant 1$ for which

$$\nu(a_n) = \nu(a_{n+k}) \quad \text{for all } k \geqslant 0.$$

For $k \geqslant 2$, we have

$$\nu(a_{n+k}) = \nu(a_{n+k-2}) = \nu(\phi^2(a_{n+k})),$$

and so by Lemma 2,

$$a_{n+k} = 2^{e_k} p_k \quad \text{for } k \geqslant 2,$$

where $e_k \geqslant 1$ and p_k is a prime with $p_k \equiv 3 \pmod 4$. Now

$$2^{e_{k-1}} p_{k-1} = a_{n+k-1} = \phi(a_{n+k}) = 2^{e_k}\frac{p_k-1}{2},$$

from which we conclude that

$$\frac{p_k-1}{2} = p_{k-1}.$$

Thus, $p_k = 2p_{k-1}+1$, and it follows from induction that

(1) $$p_k = 2^{k-2}(p_2+1)-1 \quad \text{for } k \geqslant 2.$$

Since the p_k are odd primes, put $k = p_2+1$ in (1) so that

$$p_k = 2^{p_2-1}(p_2+1)-1 \equiv 0 \pmod{p_2},$$

so then $p_k = p_2$, a clear contradiction to (1). This completes the proof of the lemma and the theorem.

Reference

1. H. Shapiro, An arithmetic function arising from the ϕ function, this MONTHLY, 50 (1943) 18–30.